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CONTENTS

BOUNDED FUNCTIONS OF TWO COMPLEX VARIABLES.*

By Stefan Bergman and Menahem Schiffer.

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1. Introduction. The space $\mathfrak{C}^4$ of two complex variables differs from the four-dimensional Euclidean space $\mathfrak{E}^4$ in that manifolds of certain type are distinguished in $\mathfrak{C}^4$ by the special behavior of analytic functions of two complex variables in them. (See B_0.[1]) Such manifolds are, e. g., analytic surfaces and analytic hypersurfaces, or segments of these manifolds. (See **2.**)

In this paper we consider a four-dimensional domain [2] $\mathfrak{M}^4$. We assume that a segment, $\mathfrak{i}^3$, of an analytic hypersurface, belongs to the boundary of $\mathfrak{M}^4$. Let $\{f(z_1, z_2)\}$ be a family of analytic functions, uniformly bounded on $\mathfrak{M}^4 + \mathfrak{i}^3$. In the present paper we show that under certain additional hypotheses the functions f, form a normal family not only on $\mathfrak{M}^4$ but on $\mathfrak{M}^4 + \mathfrak{i}^3$.

2. Lemmas. In this section we shall prove two lemmas concerning families of analytic and harmonic functions which depend upon a real parameter.

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[1] Various properties of $\mathfrak{C}^4$ to which we shall refer in this paper have been studied in the following papers of Bergman: B_0, Theory of pseudoconformal transformations and its connection with differential geometry, Notes of lectures delivered at Massachusetts Institute of Technology, 1939-40 (available at Brown University Library); B_1, *Mathematische Annalen*, vol. 109 (1934), p. 324; B_2, *Mathematische Zeitschrift*, vol. 39 (1934), p. 76 and 605; B_3, *Math. Sbornik*, vol. 1 (43) (1936), p. 851; B_4, *Mathematische Annalen*, vol. 104 (1931), p. 611; B_5, Sur les fonctions orthogonales . . ., Interscience Publishers, New York, 1941; B_6, *American Journal of Mathematics*, vol. 63 (1941), p. 295.

We shall refer to these papers by " B_k."

[2] We shall denote manifolds by Gothic letters, the upper index showing the dimension of the manifold. The symbols S, $+$ (sum set), $\cdot$ (intersection) will be used in the usual way. Thus we shall denote the sum of a sequence of sets, $\mathfrak{C}^n(a)$, depending on a parameter a which ranges over a set $\mathfrak{R}^m$, by $\underset{a \in \mathfrak{R}^m}{S} \mathfrak{C}^n(a)$. (Note that the sets $\mathfrak{C}^n(a)$ considered in this paper are often families of disjoint manifolds lying in the four-dimensional space so that $\underset{a \in \mathfrak{R}^m}{S} \mathfrak{C}^n(a)$ has the dimension $m + n$.) The boundary of a manifold is generally denoted by the same letter; e. g. $\mathfrak{m}^3$ means the boundary of $\mathfrak{M}^4$, $\mathfrak{b}^1(\lambda)$ the boundary of $\mathfrak{B}^2(\lambda)$, etc. A bar over a symbol denoting an open manifold indicates that the manifold is to be taken together with its boundary. The intersection of a manifold, $\mathfrak{C}^n$, with the surface $g(z_1, z_2) = \text{const.}$ will be denoted by $\mathfrak{C}^n \cdot [g(z_1, z_2) = \text{const.}]$. $E[\cdot \cdot \cdot]$ is understood to be the set of all points which satisfy the conditions mentioned in the brackets.

LEMMA I. *Let $F_n(Z, \lambda)$, $(n = 1, 2, \cdots)$ be a set of functions defined in $\mathfrak{U}^3 = E[|Z| < 1, \ 0 \leqq \lambda \leqq 2\pi]$, which are uniformly bounded; i.e. there exists a constant A, such that $|F_n(Z, \lambda)| \leqq A$, $(Z, \lambda) \ \varepsilon \ \mathfrak{U}^3$. For every fixed λ, the [3] $F_n(Z, \lambda)$ are analytic functions of Z, $|Z| < 1$. Finally they are uniformly continuous in λ in the following sense: for every $r < 1$ and $\epsilon > 0$ there exists a $\delta(\epsilon, r)$ such that*

(1) $$\lim_{\epsilon \to 0} \delta(\epsilon, r) = 0,$$

and furthermore

(2) $$|F_n(Z, \lambda_1) - F_n(Z, \lambda_2)| \leqq \epsilon$$

for $|\lambda_1 - \lambda_2| \leqq \delta(\epsilon, r)$ and $|Z| \leqq r < 1$. Under these assumptions the $F_n(Z, \lambda)$ form a normal family in $\mathfrak{U}^3$.

Proof. For each $|Z| < 1$

(3) $$F_n(Z, \lambda) = (2\pi i)^{-1} \int_{|\zeta| = 1} F_n(\zeta, \lambda) (\zeta - Z)^{-1} d\zeta$$

holds. In view of $|F_n| < A$ we have, therefore, for $|Z| \leqq r < 1$ and $|Z'| \leqq r$,

(4) $$|F_n(Z, \lambda) - F_n(Z', \lambda)| \leqq (2\pi)^{-1} \int_0^{2\pi} A |Z - Z'| (1 - r)^{-2} d\phi = B(r) |Z - Z'$$

If now a point (Z_0, λ_0), with $|Z_0| < r$, is given, there always exists a neighborhood $\boldsymbol{u}^3(Z_0, \lambda_0)$, such that for each $(Z, \lambda) \ \varepsilon \ \boldsymbol{u}^3(Z_0, \lambda_0)$ the inequality

(5) $$|F_n(Z, \lambda) - F_n(Z_0, \lambda_0)| \leqq \epsilon$$

holds. For example, $\boldsymbol{u}^3(Z_0, \lambda_0)$ can be chosen in the following way:

$$\boldsymbol{u}^3(Z_0, \lambda_0) = E\{|Z - Z_0| \leqq \text{Min}\,[\tfrac{1}{2}\epsilon[B(r)]^{-1}, r - |Z_0|],$$
$$|\lambda - \lambda_0| \leqq \text{Min}\,[\delta(\tfrac{1}{2}\epsilon, r), \lambda_0, 2\pi - \lambda_0]\}.$$

For then we have

$$|F_n(Z, \lambda) - F_n(Z_0, \lambda_0)| \leqq |F_n(Z, \lambda) - F_n(Z_0, \lambda)|$$
$$+ |F_n(Z_0, \lambda) - F_n(Z_0, \lambda_0)| \leqq \epsilon.$$

By the Heine-Borel theorem, for each given ϵ the domain $|Z| \leqq r_0 < r$, $0 \leqq \lambda \leqq 2\pi$, can be covered with a finite number of domains $\boldsymbol{u}^3(Z_v, \lambda_v)$. For λ_v fixed, the $F_n(Z, \lambda_v)$ form a normal family; therefore a partial sequence $F_{n'}(Z, \lambda_v)$ can be chosen which converges for all λ_v. Hence, there exists a number N_0 such that for $m' \geqq n' \geqq N_0$,

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[3] For the sake of brevity we often omit "$(n = 1, 2, \ldots)$" after F_n or f_n. Thus the sequence $\{f_n\}$, $(n = 1, 2, \ldots)$ is often denoted by $f_{n'}$.

(6) $$| F_{m'}(Z_v, \lambda_v) - F_{n'}(Z_v, \lambda_v) | \leqq \epsilon$$

holds at all points (Z_v, λ_v). By virtue of the choice of the points (Z_v, λ_v), the inequality

(7) $$| F_{m'}(Z, \lambda) - F_{n'}(Z, \lambda) | \leqq 3\epsilon$$

holds in the whole domain $| Z | \leqq r_0,\ 0 \leqq \lambda \leqq 2\pi$. Thus by the diagonal method we can choose a convergent partial sequence $F_r(Z, \lambda)$. In other words the $F_n(Z, \lambda)$ form a normal family in $\mathfrak{U}^3$.

LEMMA II. *Let* $U(\zeta, \lambda)$ *be a set of harmonic functions, depending on a parameter* λ, *defined in the unit circle* $\mathfrak{C}^2 = \mathrm{E}[|\zeta| < 1]$, $\zeta = re^{i\phi}$, *and uniformly bounded there:* $|U(\zeta, \lambda)| \leqq A$. *The boundary values* $u(\phi, \lambda) = U(e^{i\phi}, \lambda)$ *are supposed to be defined and continuous in the interval* $\mathfrak{j}^1 = \mathrm{E}[0 \leqq \phi \leqq 2\pi]$ *with the exception of at most* m *points* $P_v(\lambda)$ $(v = 1, 2, \cdots, m',\ m' \leqq m)$. *About each point* $P_v(\lambda)$ *we describe a circle with radius* ρ *and denote the part of it belonging to* $\mathfrak{C}^2$ *by* $\mathfrak{R}_v^2(\lambda, \rho)$. *We suppose that in the part of* $|\zeta| = 1$ *not contained in* $\mathfrak{R}_v^2(\lambda, \rho)$, *the functions* $u(\phi, \lambda)$ *are uniformly continuous with respect to* ϕ *and* λ. *Then the functions* $U(\zeta, \lambda)$ *are uniformly continuous with respect to* λ *in the closed domain* $\overline{\mathfrak{C}^2} - \overset{m'}{\underset{v=1}{\mathrm{S}}} \mathfrak{R}_v^2(\lambda, \rho_0)$ *for each fixed* $\rho_0 > 0$.

Proof.[4] Because of the relation

$$U(r'e^{i\phi'}, \lambda) - U(re^{i\phi}, \lambda) = [U(r'e^{i\phi'}, \lambda) - U(r'e^{i\phi}, \lambda)] \\ + [U(r'e^{i\phi}, \lambda) - U(re^{i\phi}, \lambda)],$$

it suffices to show that

1. $U(re^{i\phi}, \lambda)$ is continuous in $\overline{\mathfrak{C}^2} - \overset{m'}{\underset{v=1}{\mathrm{S}}} \mathfrak{R}_v^2(\lambda, \rho_0)$ as a function of ϕ, uniformly with respect to r and λ,

2. $U(re^{i\phi}, \lambda)$ is continuous in $\mathfrak{C}^2 - \overset{m'}{\underset{v=1}{\mathrm{S}}} \mathfrak{R}_v^2(\lambda, \rho_0)$ as a function of r, uniformly with respect to ϕ and λ.

1. Denoting by $\mathrm{P}(r, \phi)$ the Poisson kernel,

$$(2\pi)^{-1}(1 - r^2)(1 - 2r\cos\phi + r^2)^{-1},$$

we have

(8) $$U(re^{i\phi'}, \lambda) - U(re^{i\phi}, \lambda) = \int_0^{2\pi} [u(\phi' + \Phi, \lambda) - u(\phi + \Phi, \lambda)] \mathrm{P}(r, \Phi) d\Phi.$$

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[4] See also Bergman, B_4, pp. 617-619, where a similar method is applied.

Let $re^{i\phi}$ be a point of $\mathfrak{E}^2 - \overset{m'}{\underset{\nu=1}{S}} \mathfrak{R}_\nu{}^2(\lambda, \rho_0)$. We choose $\rho < \rho_0/2$ and denote by $\mathfrak{f}_\nu{}^1(\lambda, \rho)$ the arc of the unit circle lying in $\mathfrak{R}_\nu{}^2(\lambda, \rho)$. Then we divide the integral (8) into two parts. In the first, Φ runs through all intervals of $\mathfrak{j}^1$ where either $\exp[i(\phi' + \Phi)]$ or $\exp[i(\phi + \Phi)]$ lie in $\overset{m'}{\underset{\nu=1}{S}} \mathfrak{f}_\nu{}^1(\lambda, \rho)$; in the second, Φ runs through the remaining intervals. We may suppose that $re^{i\phi'}$ also lies in $\mathfrak{E}^2 - \overset{m'}{\underset{\nu=1}{S}} \mathfrak{R}_\nu{}^2(\lambda, \rho_0)$; therefore, in the first integral, the kernel is bounded: $|P(r, \phi)| < C(\rho_0)$. Choosing then $\rho = \text{Min}[\frac{1}{2}\rho_0, \epsilon/4mAC(\rho_0)]$, we can make the modulus of this integral less than $(\epsilon/2)$. After this choice of ϕ we make use of the uniform continuity of $u(\phi + \Phi, \lambda)$ in the remaining intervals to evaluate the second integral. If $\eta(\delta)$ is the upper bound of

$$|u(\phi' + \Phi, \lambda) - u(\phi + \Phi, \lambda)| \quad \text{for} \quad |\phi' - \phi| \leq \delta,$$

each λ and each Φ belonging to the remaining intervals, then $\lim_{\delta \to 0} \eta(\delta) = 0$. On the other hand, for $|\phi' - \phi| \leq \delta$ the formula (8) yields

$$(9) \quad |U(re^{i\phi'}, \lambda) - U(re^{i\phi}, \lambda)| \leq \eta(\delta) \int_0^{2\pi} P(r, \Phi) d\Phi + \epsilon/2 = \eta(\delta) + \epsilon/2.$$

Thus we have proved that by a proper choice of δ the difference $|U(re^{i\phi'}, \lambda) - U(re^{i\phi}, \lambda)|$ can be made less than any given ϵ and that this choice does not depend on r and λ.

2. For interior points of $\mathfrak{E}^2$ the continuity follows immediately from the integral representation in terms of the boundary values. Therefore it suffices to show that

$$(10) \quad U(re^{i\phi}, \lambda) - u(\phi, \lambda) = \int_0^{2\pi} [u(\phi + \Phi, \lambda) - u(\phi, \lambda)] P(r, \Phi) d\Phi$$

converges to 0 as $r \to 1$, uniformly for all ϕ and λ, if $e^{i\phi}$ does not lie in $\overset{m'}{\underset{\nu=1}{S}} \mathfrak{f}_\nu{}^1(\lambda, \rho_0)$. Let $\omega(\Phi)$ be the upper bound of $|u(\phi + \Phi, \lambda) - u(\phi, \lambda)|$ for all λ and all ϕ for which $e^{i\phi}$ does not lie in $\overset{m'}{\underset{\nu=1}{S}} \mathfrak{f}_\nu{}^1(\lambda, \rho_0)$. According to our hypothesis we have $\lim_{\Phi \to 0} \omega(\Phi) = 0$. Further, $\omega(\Phi)$ is continuous if Φ is sufficiently small. Thus we get from the inequality

$$(11) \qquad |U(re^{i\phi}, \lambda) - u(\phi, \lambda)| \leq \int_0^{2\pi} \omega(\Phi) P(r, \Phi) d\Phi$$

and from the convergence of $\omega(\Phi)$ to zero,

$$(12) \qquad \lim_{r \to 1} U(re^{i\phi}, \lambda) = u(\phi, \lambda),$$

uniformly with respect to ϕ and λ. This argument completes the proof of Lemma II.

3. Hypotheses. Following the developments of B_3 (see also B_5 and B_6) we recall the definition of a segment of an analytic hypersurface, and formulate some additional restrictions.

A segment of an analytic hypersurface is a three-dimensional manifold with the parametric representation

$$(13) \qquad z_\gamma = h_\gamma(Z, \lambda), \qquad\qquad (\gamma = 1, 2),$$

where the h_γ are functions of Z and λ, defined for $j^1 = E[0 \leqq \lambda \leqq 2\pi]$, $Z \varepsilon \mathfrak{B}^2(\lambda)$, having continuous derivatives with respect to λ and Z, and analytic in $\mathfrak{B}^2(\lambda)$ for each fixed λ. We assume that for each point $\{h_1(Z, \lambda), h_2(Z, \lambda)\}$, $Z \varepsilon \mathfrak{B}^2(\lambda)$, the inequalities

$$0 < |dh_v(Z, \lambda)/dZ| < \infty, \qquad 0 < |\partial(h_1, h_2)/\partial(Z, \lambda)| < \infty$$

are satisfied. Hence (13) can be solved either in the form $z_1 = h(z_2, \lambda)$ or in the form $z_2 = g(z_1, \lambda)$, with $(\partial h/\partial \lambda) \neq 0$ and $(\partial g/\partial \lambda) \neq 0$ respectively. Let us suppose for simplicity that the formula $z_1 = h(z_2, \lambda)$ holds in what follows.

If $\{Z, \lambda\}$ runs through all values of $\underset{\lambda \varepsilon j^1}{S} \mathfrak{B}_1^2(\lambda)$, the set of all points (z_1, z_2), $z_\gamma = h_\gamma(Z, \lambda)$ forms a hypersurface which can be considered as the sum $i^3 = \underset{\lambda \varepsilon j^1}{S} \mathfrak{J}_1^2(\lambda)$ of lamellae $\mathfrak{J}^2(\lambda)$, each of these lamellae being a piece of an analytic surface. We suppose that the points of each $\mathfrak{J}^2(\lambda)$ correspond in a one-to-one manner to those of $\mathfrak{B}^2(\lambda)$. Let $\mathfrak{J}^2(\lambda)$ consist of less than r connected components for all λ, a denumerable set $\mathfrak{n}^0$ of values excepted (r fixed, not depending on λ). We denote the image of $\bar{\mathfrak{B}}^2(\lambda) = \mathfrak{B}^2(\lambda) + \mathfrak{b}^1(\lambda)$ by $\bar{\mathfrak{J}}^2(\lambda)$ and we set $\bar{i}^3 = \underset{\lambda \varepsilon j^1}{S} \bar{\mathfrak{J}}^2(\lambda)$.

A point of i^3 is called a J-point if it corresponds to an interior point of $\mathfrak{B}^2(\lambda)$, and if $\lambda \between \mathfrak{n}^0$; it is called a K-point otherwise. Since we state the main theorem for the neighborhood of J-points only, we may suppose for simplicity that, for the segments of analytic hypersurfaces considered in the following, $\mathfrak{B}^2(\lambda)$ is always the circle $|Z| < 1$ and that the set $\mathfrak{n}^0$ is empty.

4. Lemmas. COROLLARY TO LEMMA [5] I. *Consider a domain $\mathfrak{M}^4$ the boundary of which contains a segment i^3 of an analytic hypersurface satis-*

[5] A preliminary report of this result has been published in the *Comptes Rendus de l'Académie des Sciences*, vol. 207 (1938), p. 711.

fying all assumptions of **3**. Let $f_n(z_1, z_2)$, $(n = 1, 2, \cdots)$, be a family of functions analytic in $\mathfrak{M}^4$, and let us denote the values $f_n(h_1(Z, \lambda), h_2(Z, \lambda))$ of f_n on $\mathfrak{i}^3$ by $F_n(Z, \lambda)$. Suppose further that the $F_n(Z, \lambda)$ satisfy the hypotheses of Lemma I. Then the functions $f_n(z_1, z_2)$ form a normal family in $\mathfrak{i}^3$.

Proof. If we write $F(Z, \lambda) = f[h_1(Z, \lambda), h_2(Z, \lambda)]$, the $F(Z, \lambda)$ satisfy the hypotheses of Lemma I. It follows by this lemma that a subset $F_n(Z, \lambda)$ can be found which converges uniformly to a limit function in every domain $|Z| \leqq \rho < 1$, $0 \leqq \lambda \leqq 2\pi$. Therefore $f_n(z_1, z_2) = f_n[h_1(Z, \lambda), h_2(Z, \lambda)]$ will converge uniformly in each closed domain $\mathfrak{i}^3$, consisting only of J-points. Hence the $f_n(z_1, z_2)$ form a normal family in $\mathfrak{i}^3$.

LEMMA III. *Suppose that the boundary* $\mathfrak{m}^3$ *of* $\mathfrak{M}^4$ *contains a segment* $\mathfrak{i}^3$ *of an analytic hypersurface of the form* $z_1 = h(z_2, \lambda)$, $z_2 \varepsilon \mathfrak{U}^2$, $\mathfrak{U}^2$ *being a simply connected domain containing* $z_2 = 0$. *Let each section* $\mathfrak{M}^4 \cdot [z_2 = t_2]$, $t_2 \varepsilon \mathfrak{U}^2$ *be a simply connected domain, the boundary of which is supposed to be a Jordan curve containing at most two K-points of* $\mathfrak{i}^3$ *and depending continuously on* t_2 *in the Fréchet sense. We suppose further that the K-points mentioned vary in a uniformly continuous way with* t_2.

Let $f_n(z_1, z_2)$ *be a family of functions which are analytic and uniformly bounded in* $\mathfrak{M}^4$, *satisfying* (2), *and converging in* $\mathfrak{M}^4 - \bar{\mathfrak{i}}^3$ *to an analytic function* $f(z_1, z_2)$, $(z_1, z_2) \varepsilon \bar{\mathfrak{M}}^4 - \bar{\mathfrak{i}}^3$. *Then this family is also normal in* $\mathfrak{M}^4 - \bar{\mathfrak{i}}^3 + \mathfrak{i}^3$, *and for each point* $(t_1, t_2) \varepsilon \mathfrak{i}^3$

$$\lim_{n' \to \infty} f_{n'}(t_1, t_2) = f(t_1, t_2) = \lim_{(z_1, z_2) \to (t_1, t_2)} f(z_1, z_2)$$

holds, $f_{n'}$ *being a conveniently chosen subsequence of* f_n.

(With regard to the significance of $\mathfrak{i}^3$, see **2**.)

Proof. Let $\zeta = g(z_1, t_2)$ be that analytic function which maps the domain $\mathfrak{D}^2(t_2) = \mathfrak{M}^4 \cdot [z_2 = t_2]$ conformally on the unit circle $|\zeta| < 1$. According to a theorem of Courant-Radó,[6] $g(z_1, t_2)$ is continuous in the closed domain $\mathfrak{P}^4 = \underset{t_2 \varepsilon \mathfrak{U}^2}{S} \mathfrak{D}^2(t_2)$, since the boundary of $\mathfrak{D}^2(t_2)$ is a Jordan curve varying continuously in the Fréchet sense with t_2. Let $\mathfrak{p}^3$ be the boundary of $\mathfrak{P}^4$; about each K-point let us describe hyperspheres $\mathfrak{H}_\gamma^4(\delta)$, with radius δ, which cut from $\mathfrak{p}^3$ a set of points $\mathfrak{q}_\delta^3$. In all remaining boundary points the sequence $f_n(z_1, z_2)$ converges uniformly to a limit function $f(z_1, z_2)$, which is uniformly continuous in $\mathfrak{p}^3 - \mathfrak{q}_\delta^3$. For, by the corollary to Lemma I, the $f_n(z_1, z_2)$

[6] R. Courant [*Nachrichten Göttingen* (1914), pp. 101-109, and (1922), pp. 69-70]. T. Radó [*Acta, Szeged*, vol. 1 (1922), pp. 180-186].

converge uniformly in the interior of $\mathfrak{i}^3$, and for the remaining part of $\mathfrak{p}^3 - \mathfrak{q}_\delta^3$ the convergence is ensured by hypothesis.

By virtue of the continuity of $g(z_1, t_2)$ in $\mathfrak{P}^4$, the points of $\mathfrak{q}_\delta^3$ are mapped, for t_2 fixed, on points within circles $\mathfrak{R}_\nu^2(t_2, \epsilon)$ with radius $\epsilon(\delta)$ and $\lim_{\delta \to 0} \epsilon(\delta) = 0$ around the points $P_\gamma(t_2)$, corresponding to the K-points. On the remaining part of $|\zeta| = 1$, $f[z_1(\zeta, t_2), t_2]$ is uniformly continuous with respect to [7] t_2.

At each interior point (z_1, z_2) of $\mathfrak{P}^4$ we define a function

$$(14) \quad f[z_1(\zeta_0, z_2), z_2] = (2\pi i)^{-1} \int_{|\zeta|=1} f[z_1(\zeta_1, z_2), z_2][(\zeta - \zeta_0)^{-1} - (\zeta - \zeta_0^{-1})^{-1}] d\zeta.$$

This equation remains correct if f is replaced by f_n. Now f_n converges uniformly to f on $|\zeta| = 1$, except for a set of arbitrarily small measure. Therefore, for all interior points of $\mathfrak{P}^4$

$$(15) \quad \lim_{n \to \infty} f_n(z_1, z_2) = f(z_1, z_2).$$

In view of the normality of the family f_n, f is an analytic function in $\mathfrak{P}^4$.

Equation (14) gives the representation of the real and imaginary part of $f(z_1, z_2) = u(z_1, z_2) + iv(z_1, z_2)$ by the Poisson integral formula. Considering $u(z_1, z_2)$ and $v(z_1, z_2)$, respectively, as families of potential functions of z_1, depending on a parameter z_2, we can apply Lemma II. If (t_1, z_2) is a point of $\mathfrak{P}^4 - \underset{\nu}{S} \mathfrak{H}_\nu^4(\delta)$, the function $f(z_1, z_2)$ converges uniformly (with respect to z_2) to $f(t_1, z_2)$ when $z_1 \to t_1$ and $(z_1, z_2) \varepsilon \mathfrak{D}^2(z_2)$. If there is a point $(t_1, t_2) \varepsilon \mathfrak{p}^3 - \mathfrak{q}_\delta^3$ and a set $\{z_1^{(\nu)}, z_2^{(\nu)}\} \varepsilon \mathfrak{P}^4 - \underset{\nu}{S} \mathfrak{H}_\nu^4(\delta)$ converging to it, we may write

$$(16) \quad |f(z_1^{(\nu)}, z_2^{(\nu)}) - f(t_1, t_2)| \leqq |f(z_1^{(\nu)}, z_2^{(\nu)}) - f(t_1^{(\nu)}, z_2^{(\nu)})| + |f(t_1^{(\nu)}, z_2^{(\nu)}) - f(t_1, t_2)|.$$

Here the sequence $t_1^{(\nu)}$ is chosen in such a way that

$$(t_1^{(\nu)}, z_2^{(\nu)}) \varepsilon \mathfrak{m}^3 \cdot [z_2 = z_2^{(\nu)}] - \mathfrak{q}_\delta^3$$

and $t_1^{(\nu)} \to t_1$ holds. This choice is possible in view of the fact that the Jordan curves $\mathfrak{m}^3 \cdot [z_2 = z_2^{(\nu)}]$ are continuous in the Fréchet sense and that the K-points, determining $\mathfrak{q}_\delta^3$, vary in a uniformly continuous way with t_2. Now the first member on the right side of (16) converges to zero in view of Lemma II; the second member converges to zero on account of Lemma I

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[7] $z_1 = z_1(\zeta, z_1^0)$ is the inverse function of $\zeta = g(z_1, z_1^0)$.

if $(t_1, t_2) \varepsilon \, \mathfrak{i}^3$, or because of the hypothesis if $(t_1, t_2) \varepsilon \, \overline{\mathfrak{M}^4} - \overline{\mathfrak{i}}^3$. This result together with (15) proves Lemma III.

5. The main theorem. *Let* $\mathfrak{M}^4$ *be a domain whose boundary contains the segment* $\mathfrak{i}^3$ *of an analytic hypersurface. Let* $f_n(z_1, z_2)$ *be a family of functions which are analytic and uniformly bounded in* $\mathfrak{M}^4 + \mathfrak{i}^3$. *If* $f_n(z_1, z_2)$ *satisfies in addition condition* (2) *then the sequence* $f_n(z_1, z_2)$ *forms a normal family in* $\mathfrak{M}^4 + \mathfrak{i}^3$, *and in each J-point* (t_1, t_2) *of* $\mathfrak{i}^3$

$$\lim_{m \to \infty} f_m(t_1, t_2) = f(t_1, t_2) = \lim_{(z_1, z_2) \to (t_1, t_2)} f(z_1, z_2)$$

holds for each convergent partial sequence of f_n.

Proof. Let us suppose for simplicity that the J-point considered is the point $(0, 0)$ and that in a certain neighborhood of it $\mathfrak{i}^3$ can be written in the

Fig. 1.[8]
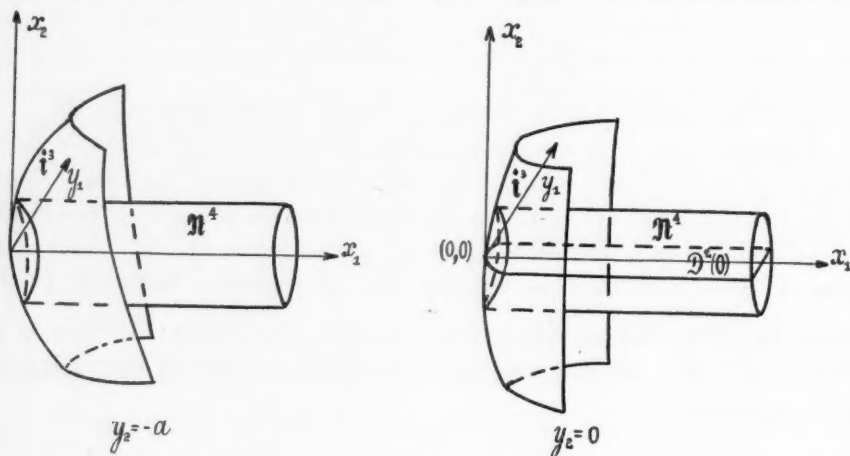
form $z_1 = h(z_2, \lambda)$. According to the developments in B_2 (p. 80), we can construct a cylinder

$$|x_1| \leqq \delta, \qquad y_1^2 + x_2^2 + y_2^2 \leqq \epsilon$$

which is divided by $\mathfrak{i}^3$ into exactly two parts. Let that part which lies inside $\mathfrak{M}^4$ be called $\mathfrak{N}^4$; further set: $\mathfrak{i}^3 \cdot \mathfrak{N}^4 = \mathfrak{h}^3$. The part $\boldsymbol{u}^3 - \mathfrak{h}^3$ of the boundary

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[8] The visualization of the four-dimensional domain carried out in our illustration is such that the $\mathrm{Im}(z_2)$ is interpreted as the time. Any given figure is the intersection of the domain under consideration with the space $\mathrm{Im}(z_2) = \mathrm{const}$. See *Jber. deutsch. Math. Ver.*, vol. 42 (1933), p. 238 and *Journal of Mathematics and Physics* (M. I. T.), vol. 20 (1941), p. 107.

of $\mathfrak{R}^4$ lies inside $\mathfrak{M}^4$; we can therefore choose a partial sequence f_r among the f_n, converging on $\boldsymbol{u}^3 - \mathfrak{i}^3$. The boundary of $\mathfrak{D}^2(t_2) = \mathfrak{R}^4 \cdot [z_2 = t_2]$, $t_2 \, \varepsilon \, \mathfrak{ll}^2$ consists of 1) an arc $z_1 = h(t_2, \lambda)$, 2) the straight lines $z_2 = t_2$, $y_1 = \pm \sqrt{\epsilon^2 - |\, t_2\, |^2}$, and 3) $x_1 = \delta$, $z_2 = t_2$. This curve is a Jordan curve varying continuously in the Fréchet sense with t_2. Lemma III is therefore applicable, and it is possible to choose a convergent sequence $f_{n'}$ which converges in $\mathfrak{R}^4 + \mathfrak{p}^3$ and which satisfies

$$(17) \qquad \lim_{n' \to \infty} f_{n'}(t_1, t_2) = f(t_1, t_2) = \lim_{(z_1, z_2) \to (t_1, t_2)} f(z_1, z_2).$$

According to the Heine-Borel theorem each interior part $\mathfrak{j}^3$ of $\mathfrak{i}^3$ can be covered by a finite number of pieces $\mathfrak{h}_\gamma{}^3$, such that each point of $\mathfrak{j}^3$ lies in the interior of at least one $\mathfrak{h}_\gamma{}^3$. The corresponding domains $\mathfrak{R}_\gamma{}^4$ cover a certain adjacent part $\mathfrak{T}^4$ of $\mathfrak{R}^4$. Using the diagonal method, we can choose a partial sequence f_ν which converges to a limit function f in this domain and satisfies (16) on the boundary. On the other hand, the f_γ also form a normal family in the remaining part of $\mathfrak{M}^4$, and since each point in $\mathfrak{M}^4$ can be joined with $\mathfrak{T}^4$, it follows by a well known argument that the sequence f_γ converges in the whole domain $\mathfrak{M}^4$. This proves our theorem.

The same result can be obtained if $\mathfrak{M}^4$ has a denumerable number of boundary hypersurfaces. If, in particular, $\mathfrak{M}^4$ is bounded only by pieces of analytic hypersurfaces, then the family of functions $f_n(z_1, z_2)$ is normal on the whole boundary, except for a two-dimensional manifold of the K-points. For these types of domains, however, this manifold forms precisely the distinguished boundary set; in this case our theorem shows that the distinguished boundary surface indeed plays the role of the boundary curve in the theory of one complex variable, while the remaining boundary points of $\mathfrak{M}^4$ are more akin to the interior points of the domain.

Under rather weaker restrictions on $\mathfrak{M}^4$ it suffices to require that the $f_n(z_1, z_2)$ are continuous and bounded in $\mathfrak{i}^3$, since such a function must be an analytic function of Z (see **2**) in each lamella. (See B_2, p. 606.)

ON NECESSARY CONDITIONS FOR RELATIVE MINIMA.*

By MARY JANE COX.

Introduction. Within recent years several papers have been published by McShane [1] on the subject of establishing, without assumptions of normality, necessary conditions and sufficient conditions for a function to have a minimum, subject to certain conditions on the independent variables. For the problems of Lagrange and Bolza in the calculus of variations, he has proved the existence of a set of multipliers with $\lambda_0 \geq 0$ for which the multiplier rule, the DuBois-Reymond relations, the transversality conditions and the analogues of the Weierstrass and Clebsch conditions all hold as necessary conditions for a minimum. He has investigated the possibility of choosing a system of multipliers with $\lambda_0 \geq 0$ for which the second variation is non-negative and has shown that, if the order of anormality is not greater than 1, the choice is possible. [2]

Likewise, under the assumption that the multiplier rule holds for a set of multipliers with $\lambda_0 \geq 0$, McShane [3] has obtained sufficient conditions for a weak relative minimum in the Bolza problem. Quite recently, using a non-negative λ_0, F. G. Myers [4] has established a sufficiency theorem for the type of minimum known as a semi-strong relative minimum.

The purpose of the present paper [5] is to establish a theorem of a rather general nature, in which is proved the existence of a set of multipliers with $\lambda_0 \geq 0$ for which not only the necessary conditions for the minimum stated above hold, but also so does the non-negativeness of the second variation. However, the multipliers are dependent, in general, on the choice of the family of comparison curves in which the minimizing curve is embedded. In the proof, use is made of the theory of convex sets, a method introduced by McShane. [6]

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[1] See, e. g., E. J. McShane, (4 to 9). The numbers in parentheses refer to the brief bibliography at the end of this paper.

[2] *Loc. cit.* (7).

[3] *Loc. cit.* (9).

[4] F. G. Myers, (11). [Added in proof: M. R. Hestenes has announced (*Bulletin of the American Mathematical Society*, vol. 49, p. 855) that he has established the corresponding sufficiency theorem for the strong relative minimum].

[6] E. J. McShane, (5, 6, 7).

1. The general problem. Throughout the paper, we shall use the summation convention on repeated indices. In this section, the range of the symbols α, β, j is as follows: $\alpha = 0, 1, \cdots, p$; β, $j = 1, 2, \cdots, p$. The parameter b denotes a finite set of numbers $b = (b_1, \cdots, b_n)$, while e represents a single number.

We assume that $f^a(z)$ are real valued functions defined on an aggregate Z of entities z, which may be called points z. There exists a point z_0 in Z such that among all the points of Z, z_0 gives a minimum to $f^0(z)$ subject to the conditions

$$(1) \qquad\qquad f^\beta(z) = 0.$$

By definition, V and W are aggregates of vectors v and w, respectively, in $(p+1)$-dimensional space having the following property: if $v_1, \cdots, v_n$ is any finite collection of vectors v of V and w is any vector of W, there exists a function $z(b_1, \cdots, b_n, e)$ defined on the set

$$(2) \qquad 0 \leqq b_k \leqq h_k; \ 0 \leqq e \leqq h; \ h, h_k > 0 \qquad\qquad (k = 1, \cdots, n)$$

such that $f^a(z(b, e))$ is of class C^2 on the set (2) and, for $b = e = 0$, the following relations hold:

$$(3) \qquad\qquad z(0, 0) = z_0,$$

$$(4) \qquad\qquad \partial f^a(z(0,0))/\partial b_k = v_k{}^a \qquad\qquad (k = 1, \cdots, n),$$

$$(5) \qquad\qquad \partial f^0(z(0,0))/\partial e \leqq 0,$$

$$(6) \qquad\qquad \partial f^\beta(z(0,0))/\partial e = 0,$$

$$(7) \qquad\qquad \partial^2 f^a(z(0,0))/\partial e^2 = w^a.$$

Let the set V^* be defined as the set of all vectors $\bar{v}$ of the form $a_1 v_1 + \cdots + a_m v_m$, $a_i \geqq 0$, v_i in V, $(i = 1, \cdots, m)$. Evidently V^* contains V. Furthermore V^* has the properties of V postulated above. For let $\bar{v}_1, \cdots, \bar{v}_n$ be any finite collection of vectors of V^* and let w be any vector of W. Suppose that $v_1, \cdots, v_q$ are the vectors of V involved in $\bar{v}_j$, $(j = 1, \cdots, n)$. Then $\bar{v}_j = a_{jk} v_k$, $a_{jk} \geqq 0$ $(k = 1, \cdots, q)$. Corresponding to $v_1, \cdots, v_q$ and w, there exists a function $z(b_1, \cdots, b_q, e)$ possessing the properties stated in the preceding paragraph. The function

$$\bar{z}(\bar{b}_1, \cdots, \bar{b}_n, e) \equiv z(\bar{b}_j a_{j1}, \cdots, \bar{b}_j a_{jq}, e)$$

has the needed continuity properties, satisfies the relations (3), (5), (6), and (7) and furthermore

$$\frac{\partial f^a(\bar{z}(0,0))}{\partial \bar{b}_j} = \frac{\partial f^a(z(0,0))}{\partial b_k} a_{jk} = a_{jk}v_k = \bar{v}_j.$$

Hence relation (4) is satisfied likewise.

It is not difficult to prove that V^* is a convex cone with vertex at the origin.[7] Henceforward, we shall use the more general set V^* as our set V.

We now define an aggregate V^+ as the set of all vectors u such that u in V^+ implies the existence of a vector v in V satisfying the relation,

$$u = v + \epsilon\delta_0, \ \epsilon \geqq 0, \ \delta_0 = (1, 0, \cdot \cdot \cdot, 0).$$

It is easily shown that $\overline{V^+}$ is a closed convex cone with vertex at the origin.

We shall establish the following theorem:

THEOREM I. *If $f^0(z)$ has a minimum on Z at z_0 subject to the conditions $f^\beta(z) = 0$ then for each vector w in W, there exist numbers $l_0 \geqq 0, l_1, \cdot \cdot \cdot, l_p$, not all zero such that for every vector v in V it is true that*

(8) $l_a v^a \geqq 0$

and also

(9) $l_a w^a \geqq 0.$

The essential part of the proof is contained in the lemma which follows:

LEMMA. *The vector $-w$ is not interior to $\overline{V^+}$, the closure of the convex set V^+.*

Suppose the statement to be false. Then, $-w$, being interior to $\overline{V^+}$, is necessarily interior to V^+. Hence for a sufficiently small positive number η, it is possible to find a vector $\bar{u}$, also interior to V^+, such that

(10) $-w = \bar{u} + \eta\delta_0, \ w^0 \neq \eta > 0;$

that is,

(11) $\bar{u}^0 + w^0 < 0, \ \bar{u}^\beta = -w^\beta, \ \bar{u}^0 \neq 0.$

If a sufficiently small positive number δ is chosen, the vectors

$$u_1 = \bar{u} + (0, \delta, 0, \cdot \cdot \cdot, 0)$$
$$u_2 = \bar{u} + (0, 0, \delta, \cdot \cdot \cdot, 0)$$

(12)
$$\cdot$$
$$\cdot$$
$$\cdot$$

$$u_p = \bar{u} + (0, 0, \cdot \cdot \cdot, \delta)$$
$$u_{p+1} = \bar{u} + (0, -\delta, -\delta, \cdot \cdot \cdot, -\delta)$$

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[7] E. J. McShane, (5, 6).

are all interior to V^+. It is evident that

$$(13) \qquad \sum_{k=1}^{p+1} u_k = (p+1)\bar{u}.$$

Now since each u_k is interior to V^+ there exist vectors v_k in V and numbers $\gamma_k \geqq 0$ such that

$$(14) \qquad u_k = v_k + \gamma_k \delta_0 \qquad\qquad (k = 1, \cdots, p+1).$$

From (11), (13), and (14) we obtain

$$(15) \qquad [1/(p+1)] \sum_{k=1}^{p+1} v_k^0 = \bar{u}^0 - [1/(p+1)] \sum_{k=1}^{p+1} \gamma_k$$

and

$$(16) \qquad [1/(p+1)] \sum_{k=1}^{p+1} v_k^\beta = \bar{u}^\beta = -w^\beta.$$

Referring to the definition of the sets V and W, we see that, corresponding to the vectors v_k and w, there exists a function $z(b_1, \cdots, b_{p+1}, e)$, defined on the interval (2), the values of which represent points of Z reducing to z_0 for $b = e = 0$. Furthermore $z(b, e)$ defines functions $f^a(z(b, e))$ of class C^2 on (2) such that their first and second derivatives $f^a{}_{b_k}, f^a{}_e, f^a{}_{ee}$ satisfy the relations (4), (5), (6), and (7) with w and the vectors v_k of (14). The functions $\phi^a(b, e) \equiv f^a(z(b, e))$ are of class C^2 on the set (2) and can be extended [8] to be of class C^2 on $-h_k \leqq b_k \leqq h_k, -h \leqq e \leqq h; h, h_k > 0, (k = 1, \cdots, p+1)$. However, by the definition of V, the values of $\phi^a(b, e)$ can not be interpreted as values of $f^a(z(b, e))$ unless $b_k \geqq 0$ and $e \geqq 0$.

Consider the equations,

$$(17) \qquad \phi^\beta(b, e) = 0.$$

These have initial solutions $b = e = 0$ by (1). At $b = e = 0$, by (4) and (14), we see that the jacobian is

$$(18) \qquad |\phi^\beta{}_{b_j}| = |f^\beta{}_{b_j}(z_0)| = |v_j^\beta| = |u_j^\beta| \qquad (\beta, j = 1, \cdots, p).$$

This expression, being a polynomial of degree p in δ, is not identically zero.[9] Its value is not equal to zero if we choose δ, as we may, so as to avoid the zeros of the polynomial. Consequently, by the implicit function theorem, equations (17) determine the b_j uniquely as functions of b_{p+1} and e, such that the solutions $b_j = b_j(b_{p+1}, e)$ are of class C^2 near $b_{p+1} = e = 0$ and also $b_j(0, 0) = 0$.

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[8] M. R. Hestenes, (3).

[9] To be specific $|\phi^\beta{}_j(0, 0)| = \delta^{p-1}(\sum_{\beta=1}^{p} \bar{u}\beta + \delta)$, as is readily computed from (12).

If we differentiate the identities

$$(19) \qquad \phi^\beta(b_j(b_{p+1}, e), b_{p+1}, e) \equiv 0$$

with respect to e, we find that at $b = e = 0$, by virtue of (6),

$$(20) \qquad \phi^\beta_{b_j}(0, 0) \frac{\partial b_j(0, 0)}{\partial e} = 0 \qquad\qquad (\beta, j = 1, \cdots, p).$$

By the remarks following (18), this gives us

$$(21) \qquad \frac{\partial b_j(0, 0)}{\partial e} = 0 \qquad\qquad (j = 1, \cdots, p).$$

Suppose now that b_{p+1} is assigned the value $e^2/2(p+1)$ and define

$$(22) \qquad \bar{b}(e) \equiv e^2/2(p+1) = b_{p+1}, \qquad B_j(e) \equiv b_j(\bar{b}(e), e).$$

Differentiating $B_j(e)$ with respect to e, setting $e = 0$ and making use of (21) and (22), we obtain

$$(23) \qquad\qquad B_j'(0) = 0.$$

Now equations (19) have become identities in the single variable e; namely, $\phi^\beta(B_j(e), \bar{b}(e), e) \equiv 0$. As a consequence of (23), the second derivatives $d^2\phi^\beta/de^2$ reduce at $e = 0$ to

$$d^2\phi^\beta/de^2 = \phi^\beta_{b_j}(0)B_j''(0) + \phi^\beta_{b_{p+1}}(0)[1/(p+1)] + \phi^\beta_{ee}(0) = 0.$$

Referring to (4), (6) and (7), we see that this is equivalent to

$$(24) \qquad d^2\phi^\beta/de^2 = v_j^\beta B_j''(0) + v_{p+1}^\beta[1/(p+1)] + w^\beta = 0.$$

After substituting the value of w^β from (16) and collecting terms, we find that equations (24) reduce to

$$(25) \qquad v_j^\beta(B_j''(0) - c_j) = 0 \qquad\qquad (c_1 = \cdots = c_p = 1/(p+1)).$$

Since $|v_j^\beta| \neq 0$, it follows that

$$(26) \qquad B_j''(0) = c_j = 1/(p+1) > 0 \qquad\qquad (j = 1, \cdots, p).$$

From (22), (23) and (26), it is evident that $\bar{b}(e) > 0$ and $B_j(e) > 0$ for e near 0 if $0 < e \leqq h$.

Let us consider now the function $\phi^0(e) \equiv f^0(z(B_j(e), \bar{b}(e), e)$. Making use of (23) and (5), we obtain for $d\phi^0(e)/de$ at $e = 0$ the relation

$$(27) \qquad\qquad d\phi^0/de = f_e^0(z(0, 0)) \leqq 0.$$

By the method used in deriving (24), the second derivative $d^2\phi^0(e)/de^2$ at

$e = 0$ is found to be $d^2\phi^0/de^2 = v_j{}^0 B_j''(0) + v_{p+1}{}^0[1/(p+1)] + w^0$, which by the aid of (11), (15), and (26) simplifies to

$$(28) \qquad d^2\phi^0/de^2 = \bar{u}^0 + w^0 - [1(p+1)]\sum_{k=1}^{p+1}\gamma_k < 0.$$

Summing up results, we have found functions

$$\phi^a(e) \equiv f^a(z(B_j(e), \bar{b}(e), e))$$

of class C^2 on $0 \leqq e \leqq h$, $h > 0$. For all non-negative e near 0

$$\phi^\beta(e) \equiv f^\beta(z(B_j(e), \bar{b}(e), e) = 0.$$

At $e = 0$, $z(B_j(e), b(e), e) = z_0$. Therefore, $\phi^0(e)$ has a minimum at $e = 0$. Hence, at $e = 0$, the first derivative of $\phi^0(e)$ is greater than or equal to zero and, if it equals zero, then the second derivative of $\phi^0(e)$ is non-negative. The first of these inequalities together with (27) implies that

$$(29) \qquad (d\phi^0/de)|_{e=0} = 0.$$

Consequently, at $e = 0$ the second statement holds; that is,

$$(30) \qquad (d^2\phi^0/de^2)|_{e=0} \geqq 0.$$

But by (28), $d^2\phi^0/de^2 < 0$. This contradiction proves the lemma.

It is a simple matter now to prove Theorem I. By the lemma, $-w$ is either exterior to $\overline{V^+}$ or is on its boundary. If $-w$ is exterior to $\overline{V^+}$, there exists a hyperplane of support separating $-w$ from $\overline{V^+}$. Every hyperplane of support of a closed convex cone passes through the vertex (the origin in this case); hence it has the equations $l_a u^a = 0$, l_a not all zero. Thus, by changing the signs of the l_a if necessary, we have $l_a(-w^a) < 0$, while $l_a u^a \geqq 0$ for all u in $\overline{V^+}$. If $-w$ is a boundary point of $\overline{V^+}$, there is a hyperplane of support passing through $-w$; then $l_a(-w^a) = 0$, while $l_a u^a \geqq 0$ for all u in $\overline{V^+}$. In either case, it is true that $l_a w^a \geqq 0$ and $l_a u^a \geqq 0$ for all u in $\overline{V^+}$. Since $\overline{V^+}$ contains $\bar{V}$, by the definition of V^+, we obtain

$$(31) \qquad l_a w^a \geqq 0 \text{ and } l_a v^a \geqq 0 \text{ for all } v \text{ in } V.$$

Also, since the origin is in $\bar{V}$, evidently $\delta_0 = (1, 0, \cdots, 0)$ is in $\overline{V^+}$, which implies that $l_a \delta_0{}^a \geqq 0$. This reduces to $l_0 \geqq 0$. Hence Theorem I has been established.

2. The problem of Bolza. In this section we apply Theorem I to the problem of Bolza in parametric form to obtain the results stated in the intro-

duction. Throughout the discussion, the symbols i, j, h, β, γ will have the ranges, $i = (1, \cdot \cdot \cdot, n)$, $j = (1, \cdot \cdot \cdot, r)$, $h = (1, \cdot \cdot \cdot, l)$, $\beta = (1, \cdot \cdot \cdot, m < n - 1)$, $\gamma = (m + 1, \cdot \cdot \cdot, n)$. The prime $(')$ is used to denote differentiation with respect to the variable t.

The problem to be considered is the following: To minimize the functional

$$(32) \qquad J(C, \alpha) = \theta(\alpha) + \int_{t_1}^{t_2} f(y(t), y'(t)) \, dt$$

on the class of admissible sets (C, α)[10] satisfying the differential equations

$$(33) \qquad \phi^\beta(y(t), y'(t)) = 0 \qquad\qquad (\beta = 1, \cdot \cdot \cdot, m < n - 1)$$

and the end conditions

$$(34) \qquad y^i(t_s) = T_s^i(\alpha) \qquad\qquad (i = 1, \cdot \cdot \cdot, n; \ s = 1, 2).$$

For brevity we shall use the alternative notations; $I(C)$ for the integral in (32), and y_s^i for $y^i(t_s)$ when referring to the end points of an admissible curve.

As usual we make the following assumptions: R_1 is a region in a $2n$-dimensional space of points $(y, r) = (y^1, \cdot \cdot \cdot, y^n, r^1, \cdot \cdot \cdot, r^n)$ having the property that for any (y, r) in R_1 and any number $k > 0$, (y, kr) is also in R_1. The functions $f(y, r)$ and $\phi^\beta(y, r)$ are defined and of class C^2 for (y, r) in R_1, $|r| \neq 0$, and are positively homogeneous of degree 1 in r. R_2 is a region of points $\alpha = (\alpha^1, \cdot \cdot \cdot, \alpha^r)$ in an r-dimensional space which contains the origin and on which the functions $\theta(\alpha)$ and $T_s^i(\alpha)$ are defined and of class C^2.

By definition, a set (C, α) consisting of a curve of class D^1,

$$C: \quad y^i = y^i(t) \qquad\qquad (t_1 \leqq t \leqq t_2)$$

and an r-tuple (α) is admissible if each $(y(t), y'(t))$ on C is interior to R_1, satisfies the differential equations (33) and the matrix $\| \phi^\beta{}_{r^i} \|$ has rank m on $[t_1, t_2]$, and the point (α) is interior to R_2.

We suppose that the set $(C_0, 0)$ consisting of the curve $C_0 \colon y^i = y_0^i(t)$ and the r-tuple $(\alpha) = (0, \cdot \cdot \cdot, 0)$ is admissible. As is well known, if C_0 is embedded in a family of admissible curves $y^i = y^i(t, b)$ reducing to C_0 for $b = 0$, then the *variations of the family along* C_0,

$$(35) \qquad\qquad \eta^i(t) = y_b^i(t, 0),$$

must satisfy the *equations of variation of the side conditions* (33), namely

$$(36) \qquad\qquad \Phi^\beta(\eta, t, \eta') = 0$$

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[10] These sets will be defined later.

where by definition

$$\Phi^\beta(\eta, t, \rho) = \phi^\beta_{y^i}(y_0(t), y_0'(t))\eta^i + \phi^\beta_{r^i}(y_0(t), y_0'(t))\rho^i.$$

It is likewise well known that given a set of functions $\eta_1, \cdots, \eta_l$ of class D^1 on $[t_1, t_2]$, satisfying equations (36) with a non-singular matrix $\| \phi^\beta_{r^i} \|$, there exists a family of D^1 admissible curves embedding C_0 such that the variations of the family along C_0 are identical with the given functions $\eta_h(t)$, $(h = 1, \cdots, l)$. In establishing this theorem, Bliss [11] used a device which we shall need in the sequel and which has been adapted to the parametric problem of Bolza by F. G. Myers,[12] as summarized in Lemmas 1 and 2 below.

LEMMA 1. *If the admissible curve C_0 is of class C^1, there exist functions $\phi^\gamma(t, r)$, $(t_1 \leqq t \leqq t_2;$ all $r; \gamma = m + 1, \cdots, n)$ of whatever class desired such that the determinant*

$$(37) \qquad \begin{vmatrix} \phi^\beta_{r^i}(y_0(t), y_0'(t)) \\ \phi^\gamma_{r^i}(t, y_0'(t)) \end{vmatrix} \neq 0 \qquad (t_1 \leqq t \leqq t_2).$$

By definition, the functions $\phi^\gamma(t, r)$ are chosen to satisfy the relations

$$\phi^\gamma(t, r) \equiv c_i(t)r^i$$

where $c_i(t)$ are polynomials which approximate as closely as desired continuous functions of t, the existence of which has been proved by Bliss.[13] It is not difficult to show that Bliss' lemma holds for functions $\phi^\beta_{r^i}(y_0, y_0')$ of class D^0 and hence the lemma holds for a curve C_0 of class D^1.

Now define

$$\Phi^\gamma(\eta, t, \rho) \equiv \phi^\gamma_{r^i}(t, y_0'(t))\rho^i = c_i(t)\rho^i.$$

Any set of functions $\eta^i(t)$ of class D^1 determines uniquely a set of functions $\zeta^i(t)$ of class D^0 by the transformation

$$(38) \qquad \zeta^i(t) = \Phi^i(\eta, t, \eta').$$

The $\eta^i(t)$ satisfy the equations (36) if and only if $\zeta^1(t) = \cdots = \zeta^m(t) = 0$. The $\zeta^\gamma(t)$ are continuous except possibly at corners of C_0 and at points of discontinuity of the derivatives $\eta^{i'}(t)$.

LEMMA 2. *Let $\zeta^{m+1}(t), \cdots, \zeta^n(t)$ be functions of class D^0 on $[t_1, t_2]$ and let $d^1, \cdots, d^n$ be numbers representing the values of the $\eta^i(t)$ at any one point t_0 on $[t_1, t_2]$. Then since (37) holds, we can solve the equations,*

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[11] G. A. Bliss, (1).   [12] F. G. Myers, (10).   [13] *Loc. cit.*, (1).

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$$\phi^{\beta}_{y_i}(y_0(t), y_0'(t))\eta^i + \phi^{\beta}_{r^i}(y_0(t), y_0'(t))\eta^{i'} = 0,$$
$$\phi^{\gamma}_{r^i}(t, y_0'(t))\eta^{i'} = \zeta^{\gamma}$$

for $\eta^{i'}$ as functions of η^i and ζ^{γ},

$$\eta^{i'} = \rho^i(\eta, \zeta).$$

Since by hypothesis the $\zeta^{\gamma}(t)$ are known functions of class D^0 on $[t_1, t_2]$, the differential equations

$$\eta^{i'}(t) = \rho^i(\eta(t), \zeta(t))$$

have a unique solution $\eta(t)$ of class D^1 on $[t_1, t_2]$ satisfying the initial conditions

$$\eta^i(t_0) = d^i.$$

It should be noted that the above lemmas are valid if instead of a single set $\eta^i(t)$ we have a finite number of sets $\eta_1{}^i(t), \cdots, \eta_l{}^i(t)$ provided the variations $\eta_h{}^i(t)$, $(h = 1, \cdots, l)$, are interpreted to be the derivatives $y_{b_h}{}^i(t, 0)$ where $y^i = y^i(t, b_1, \cdots, b_l)$ is an l-parametered family of admissible curves containing C_0 for $b_1 = \cdots = b_l = 0$.

For future reference we shall need the following lemma for the proof of which we are indebted to McShane.[14]

LEMMA 3. *Let the element $(y_0(t_0), r_0)$, $r_0 \neq 0$, $t_1 \leq t_0 \leq t_2$, be interior to R_1 and satisfy the side conditions (33) with matrix $\| \phi^{\beta}_{r^i}(y_0(t_0), r_0 \|$ of rank m. Then there exists an $\epsilon > 0$, a neighborhood $N_{\epsilon}(y_0(t_0))$ and a family of curves*

$$y^i = Y^i(\tau, \bar{y}),$$

defined and of class C^2 for $-\epsilon \leq \tau \leq \epsilon$ and $\bar{y}$ in $N_{\epsilon}(y_0(t_0))$, with the properties;

(39-i)   *τ is the arc-length on $Y(\tau, \bar{y})$ measured from $\bar{y}$,*

ii)   *$Y^i(0, \bar{y}) = \bar{y}$,*

iii)   *$\dfrac{d}{d\tau} Y^i(0, y_0(t_0)) = r_0{}^i/|r_0|$,*

iv)   *$\phi^{\beta}(Y, Y') \equiv 0$ in τ and $\bar{y}$ $(|\tau| \leq \epsilon, \bar{y}$ in $N_{\epsilon}(y_0(t_0)))$,*

v)   *The curves of the family simply cover $N_{\epsilon}(y_0(t_0))$.*

Since the matrix $\| \phi^{\beta}_{r^i}(y_0(t_0), r_0) \|$ has rank m, it is possible to adjoin

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[14] E. J. McShane, Lectures on the problem of Bolza delivered at the University of Virginia, 1941-42.

constants $c_i{}^\gamma$ so that the determinant $\left| \begin{matrix} \phi^\beta{}_{r^i}(y_0(t_0), r_0) \\ c_i{}^\gamma \end{matrix} \right|$ is not zero. The proof of the lemma follows by an application of the implicit function theorem to obtain solutions $r^i = r^i(y)$ of the equations

(40) $$\phi^\beta(y, r) = 0, \qquad c_i{}^\gamma r^i = c_i{}^\gamma r_0{}^i$$

and then by use of the existence theorems for differential equations to get a solution of the equations

$$y^{i'}(\tau) = \frac{r^i(y)}{[r^i(y) \cdot r^i(y)]^{\frac{1}{2}}},$$

which is possible since the functions $r^i(y)$ are not zero on $N_\epsilon(y_0(t_0))$ for ϵ small enough. For brevity we omit the details of the proof.

We turn now to the construction of a family of D^1 curves containing C_0 and having the properties needed for our problem. The method used is a modification of that introduced by McShane.[15]

Let the vector function $\eta(t) = (\eta^1(t), \cdots, \eta^n(t))$ be of class D^1 on $[t_1, t_2]$ except that it may have a single jump (finite) discontinuity on this interval subject to the condition: if $\eta(t)$ has a jump r at $\bar{t}$, then $(y_0(\bar{t}), r)$ is interior to R_1. By definition, a function $\eta(t)$ of the type described above is called an *admissible variation along* C_0 provided the following conditions hold:

(41-i)   *the functions* $\eta^i(t)$ *satisfy the equations of variation* (36) *and*

ii)   *the components of the element* $(y_0(\bar{t}), r)$ *satisfy the differential equations* (33) *with matrix* $\| \phi^\beta{}_{r^i}(y_0(\bar{t}), r) \|$ *of rank* m.

Throughout the remainder of the paper we shall assume that the conditions (41-i, ii) hold.

We shall prove the following lemma, adapting to our needs methods used by Bliss and McShane.[16]

EMBEDDING LEMMA. *Let* $e = (e_1, \cdots, e_q)$ *be a multiple such that there exists a set of functions* $y^i(t, e)$ *continuous on* $t_1 \leqq t \leqq t_2$, $0 \leqq e_i \leqq h_i$, *having the properties that* y^i *are of class* D^1 *in* t *for each fixed* e *and* y^i *and* $y^{i'}$ *are of class* C^2 *in* e *for each fixed* t. *Let the equations*

(42) $$y^i(t, 0) \equiv y_0{}^i(t) \qquad\qquad (t_1 \leqq t \leqq t_2)$$
and
(43) $$\phi^\beta(y(t, e), y'(t, e)) = 0 \qquad\qquad (t_1 \leqq t \leqq t_2, \ 0 \leqq e_i \leqq h_i)$$

hold. Let $\eta_1(t), \cdots, \eta_l(t)$ *be admissible variations having respective jumps* $r_1, \cdots, r_l$ *at* $\bar{t}_1, \cdots, \bar{t}_l$ *in* $[t_1, t_2]$.

[15] E. J. McShane, (6).

[16] G. A. Bliss, (1, pp. 15-19, 58-61; 2, pp. 678-679); E. J. McShane, (6, pp. 810-811).

Then there exists a family of curves

$$C(e, b) \equiv C(e, b_1, \cdots, b_l),$$

defined for all small non-negative (e, b), with the following properties:

(44-i)   *Each $C(e, b)$ is of class D^1 on $[t_1, t_2]$ and satisfies the equations*

$$\phi^\beta(y, y') = 0 \qquad\qquad (\beta = 1, \cdots, m).$$

ii)   *For each e on $[0, h]$ the curve $C(e, 0)$ is the same as the curve*

$$y^i = y^i(t, e) \qquad\qquad (t_1 \leqq t \leqq t_2).$$

iii)   *The integral $I(C(e, b))$ evaluated along $C(e, b)$ is of class C^2 as a function of (e, b) and so are the coördinates of the end-points of $C(e, b)$.*

iv)   *For each $h = 1, \cdots, l$ the equation*

$$\frac{\partial}{\partial b_h}[I(C(e, b))]_{e=b=0} = \int_{t_1}^{t_2} [f_{y^i}(y_0(t), y_0'(t))\eta_h{}^i(t)$$
$$+ f_{r^i}(y_0(t), y_0'(t))\eta_h{}^{i'}(t)]dt + f(y_0(\bar{t}_h), r_h)$$

is satisfied.

We consider first a single admissible variation, continuous on $[t_1, t_2]$. With the aid of Lemma 1, we adjoin polynomials $c_i{}^\gamma(t)$ to the functions $\phi^\beta{}_{r^i}(y_0(t), y_0'(t))$ so that the determinant (37) is non-singular along C_0. By the transformation (38), the variation $\eta(t)$ determines uniquely a set of functions $\zeta^\gamma(t)$ of class D^0 on $[t_1, t_2]$, continuous between points of discontinuity of the derivatives $y_0^{i'}$, $\eta^{i'}$, which together with $\eta(t)$ satisfy on that interval the equations,

(45) $\Phi^\beta(\eta, t, \eta') \equiv \phi^\beta{}_{y^i}(y_0(t), y_0'(t))\eta^i(t) + \phi^\beta{}_{r^i}(y_0(t), y_0'(t))\eta^{i'}(t) = 0$

$\Phi^\gamma(\eta, t, \eta') \equiv \phi^\gamma{}_{r^i}(t, y_0'(t))\eta^{i'}(t) = \zeta^\gamma(t).$

From the continuity properties of the functions involved in (37), it follows that there exists an $\epsilon > 0$ and less than every h_i such that this determinant remains non-singular along each curve of the family

(46) $y^i = y^i(t, e)$ $\qquad\qquad (t_1 \leqq t \leqq t_2,\ 0 \leqq e_i \leqq \epsilon \leqq h_i)$

defined by the functions in the hypothesis. Hence Bolza's form of the implicit function theorem and the existence theorems for differential equations tell us that the system of equations

(47) $\phi^\beta(y, r) = 0, \qquad \phi^\gamma(t, r) = \phi^\gamma(t, y'(t, e)) + b\zeta^\gamma(t)$

determines uniquely a family of solutions

$$(48) \qquad y^i = y^i(t, e, b). \qquad\qquad (t_1 \leqq t \leqq t_2,\ 0 \leqq e_i \leqq \epsilon,\ b\ near\ 0)$$

with the initial conditions

$$(49) \qquad y_1{}^i(e, b) \equiv y^i(t_1, e, b) = y^i(t_1, e) + b\eta^i(t_1).$$

The solutions (48) possess the following properties:

(50-i) *For each fixed set (e, b) the functions $y^i(t, e, b)$ are of class D^1 in t*
(C^1 between corners of $y_0{}^i(t)$, $y^i(t, e)$, or $\eta^i(t)$); the y^i together
with their derivatives $y^{i\prime}$ are of class C^2 in e and b for each fixed t,
hence the end-functions $y_s{}^i(e, b)$, $(s = 1, 2)$, are of class C^2.

ii) *The family (48) satisfies the equations*

$$\phi^\beta(y(t, e, b), y'(t, e, b)) \equiv 0$$

identically in t, e, and b. At $b = 0$ it reduces to the given family
(46), that is

$$y^i(t, e, 0) = y^i(t, e) \qquad\qquad (t_1 \leqq t \leqq t_2,\ 0 \leqq e_i \leqq \epsilon)$$

and hence by (42) at $e = b = 0$ to the curve

$$y^i = y_0{}^i(t) \qquad\qquad (t_1 \leqq t \leqq t_2).$$

iii) *The derivatives $y_b{}^i(t, 0, 0)$ satisfy the equations (45), thus defining*
the same set $\zeta^\gamma(t)$ on $[t_1, t_2]$ as do the given functions $\eta^i(t)$. Since
by (49) they satisfy the initial conditions

$$y_{1,b}{}^i(0, 0) \equiv y_b{}^i(t_1, 0, 0) = \eta^i(t_1),$$

it follows from Lemma 2 that the equations

$$y_b{}^i(t, 0, 0) = \eta^i(t) \qquad\qquad (t_1 \leqq t \leqq t_2)$$

hold.

Next we consider a single admissible variation $\eta(t)$ having a jump $r_0 \neq 0$ at t_0 interior to $[t_1, t_2]$. For simplicity we suppose at first that r_0 has norm $|r_0| = 1$, a restriction which we remove later. Redefine the curve family $y^i = y^i(t, e)$, $(t_1 \leqq t \leqq t_2)$, thus;

$$(51) \qquad\begin{aligned} y^i &= \bar{y}^i(t, e) \equiv y^i(t, e) & (t_1 \leqq t \leqq t_0), \\ y^i &= \bar{y}^i(t, e) \equiv y^i(t_0, e) & (t_0 < t < t_0 + 1), \\ y^i &= \bar{y}^i(t, e) \equiv y^i(t - 1, e) & (t_0 + 1 \leqq t \leqq t_2 + 1). \end{aligned}$$

The hypotheses still hold.

On $[t_1, t_0]$ the variation $\eta(t)$ is continuous, so that the preceding argument applies. Thus we get the family

(52) $y^i = y^i(t, e, b)$ $(t_1 \leqq t \leqq t_0,\ 0 \leqq e_i \leqq \epsilon,\ b \text{ near } 0)$

satisfying the initial conditions (49) and possessing on $[t_1, t_0]$ the properties (50). For convenience we write the end-functions $y^i(t_0, e, b)$, which are of class C^2 in e and b, in the form

(53) $y^i(t_0, e, b) = y^i(t_0, e) + b\eta^i(t_0 -) + o(b)$

where $o(b)$ is a function of b which vanishes to a higher order than $|\,b\,|$.

Now if e and b are small enough, every point on an arc of the family (52) will lie arbitrarily near a point of C_0: consequently the point $y(t_0, e, b)$ will be interior to the neighborhood $N_\epsilon(y_0(t_0))$ associated with the element $(y_0(t_0), r_0)$ by Lemma 3. Thus for a sufficiently small positive b, say $0 < b \leqq \epsilon_1$, we have an arc

(54) $y^i = Y^i(\tau, y(t_0, e, b))$ $(0 \leqq \tau \leqq b)$

satisfying the initial conditions

(55) $Y^i(0, y(t_0, e, b)) = y^i(t_0, e, b)$

and having the properties (39, i-v). In particular the equations

(56) $Y_\tau^i(0, y_0(t_0)) = r_0^i$

hold at $e = b = 0$.

Next we map τ linearly on $[t_0, t_0 + 1]$, thus

$$t = t_0 + \tau/b, \qquad \tau = b(t - t_0),$$

and define

(57) $y^i(t, e, b) = Y^i(b(t - t_0), y(t_0, e, b))$ $(t_0 \leqq t \leqq t_0 + 1)$,

thereby obtaining a family of arcs

(58) $y^i = y^i(t, e, b)$ $(t_0 \leqq t \leqq t_0 + 1,\ 0 \leqq e_i \leqq \epsilon,\ 0 \leqq b \leqq \epsilon_1)$

joining continuously onto the family (52) at the point $y(t_0, e, b)$. Recalling the homogeneity property of the functions $\phi^\beta(y, y')$ and making use of the relations (39-iv), (51), (53), and (55), we see that the arcs in (58) satisfy the conditions (50-i, ii) on $[t_0, t_0 + 1]$. The end-functions $y^i(t_0 + 1, e, b)$, of class C^2 in e and b, may be written with the help of (53-55) in the form

(59) $y^i(t_0 + 1, e, b) = y^i(t_0, e) + b\eta^i(t_0 -) + bY_\tau^i(0, y(t_0, e, b)) + o(b)$.

We transpose $y^i(t_0, e)$ to the left member of (59), divide by b, and evaluate the limits of the quotients for $e = b = 0$, using the equations (51) and (56). This yields the equations

(60) $y_b^i(t_0 + 1, 0, 0) = \eta^i(t_0 -) + r_0^i = \eta^i(t_0 +)$.

Now since the variation $\eta(t)$ is continuous on $[t_0, t_2]$, we apply again the first part of the proof, obtaining a family of arcs

$$(61) \qquad y^i = y^i(t, e, b) \qquad (t_0 + 1 \leqq t \leqq t_2 + 1, \; 0 \leqq e_i \leqq \epsilon, \; 0 \leqq b \leqq \epsilon_1)$$

joining continuously onto the family (58) at the point $y(t_0 + 1, e, b)$ and possessing the properties (50) on $[t_0 + 1, t_2 + 1]$, as is readily verified with the aid of (51), (57), (60) and Lemma 2.

Thus the three families (48), (58) and (61) together form a family of continuous curves

$$(62) \qquad y^i = y^i(t, e, b) \qquad (t_1 \leqq t \leqq t_2 + 1, \; 0 \leqq e_i \leqq \epsilon, \; 0 \leqq b \leqq \epsilon_1)$$

of class D^1 in t and such that each (y, y') on (62) is in R_1 if e and b are small enough.

The functions $y^i(t, e, b)$ in (62) have all the properties demanded of $y^i(t, e)$ in the hypothesis, so that the process can be repeated. Thus the restriction to one variation η at a time is no restriction. Hence if we prove (44, i-iv) for the single η, the conclusion will hold for any finite number.

The statements in (44, i-ii) have been established in the process of constructing the family (62). Likewise we have shown that the end-functions $y_s^i(e, b)$, $(s = 1, 2)$, are of class C^2 in e and b. Along a curve of the family, the integral I has the value

$$(63) \qquad \int_{t_1}^{t_2+1} f(y(t, e, b), y'(t, e, b)) \, dt$$
$$= \int_{t_1}^{t_0} f(y, y') \, dt + \int_{t_0}^{t_0+1} f(Y, Y') \, dt + \int_{t_0+1}^{t_2+1} f(y, y') \, dt$$

where the arguments in the second integral on the right are $(b(t - t_0),$ $y(t_0, e, b))$ and those in the first and third integrals are (t, e, b). The verification of (44-iii) is easy and so we omit the details. As a result of the homogeneity property of $f(y, y')$, the second integral is invariant under the change of parameter from t to τ defined by $t = t_0 + \tau/b$, thus it is equal to the integral

$$\int_0^b f(Y(\tau, y(t_0, e, b)), Y_\tau(\tau, y(t_0, e, b))) \, d\tau.$$

After substituting this in (63), a simple computation yields the equation

$$(64) \qquad \frac{\partial}{\partial b} \int_{t_1}^{t_2+1} f(y(t, e, b), y'(t, e, b)) \, dt \, |_{e=b=0}$$
$$= \int_{t_1}^{t_0} [f_{y^i}(y_0(t), y'_0(t))\eta^i(t) + f_{r^i}(y_0(t), y_0'(t))\eta^{i'}(t)] \, dt + f(y_0(t_0), r_0).$$

Thus we have established (44-iv) and completed the proof for the single variation. Consequently, by the argument made above, the lemma holds for any finite number of variations.

The assumption that $|\, r_0\, | = 1$, made early in the proof, is no restriction. For if $|\, r_0\, | \neq 1$ the proof is still valid if we replace η by $\bar{\eta} = \eta /\, |\, r_0\, |$ and the paramter b by $\bar{b} = |\, r_0\, |\, b$.

COROLLARY. *If $\eta(t)$ is a continuous admissible variation, there exists a family of curves, $y^i = y^i(t, b)$, such that y^i are of class D^1 in t and y^i and $y^{i'}$ are of class C^2 in b on $[t_1, t_2]$, b near zero, and such that each element $(y(t, b), y'(t, b))$ is interior to R_1. The curves of the family satisfy the equations*

$$y^i(t, 0) = y_0{}^i(t) \qquad\qquad (t_1 \leqq t \leqq t_2),$$
$$y_b{}^i(t, 0) = \eta^i(t)$$

and

$$\phi^\beta(y(t, b), y'(t, b)) \equiv 0$$

identically in t and b, $t_1 \leqq t \leqq t_2$, b near zero.

The corollary is an immediate consequence of the preceding proof if we apply the lemma with $e_i = 0$, $l = 1$, $y^i(t, e) \equiv y_0{}^i(t)$, $\eta_1{}^i(t) = \eta^i(t)$, $r_1{}^i = 0$.

Suppose now that we are given a finite set of admissible variations $\bar{\eta}(t), \eta_1(t), \cdots, \eta_l(t)$ such that $\bar{\eta}$ is continuous on $[t_1, t_2]$ and η_h, $(h = 1, \cdots, l)$, satisfy the hypotheses of the lemma. First applying the corollary, we embed C_0 in the admissible family

$$C(e): \quad y^i = y^i(t, e)^{17} \qquad\qquad (t_1 \leqq t \leqq t_2, \; |\, e\, | \leqq \epsilon)$$

such that

$$y_e{}^i(t, 0) = \bar{\eta}^i(t) \qquad\qquad (t_1 \leqq t \leqq t_2).$$

Then applying the lemma, we embed the curves of the family $C(e)$ defined on $0 \leqq e \leqq \epsilon$ in the family

$$C(e, b): \quad y^i = y^i(t, e, b_1, \cdots, b_l) \qquad (0 \leqq e \leqq \epsilon, \; 0 \leqq b_h \leqq \epsilon_h)$$

such that

$$y_{b_h}{}^i(t, 0, 0) = r_h{}^i(t) \qquad\qquad (t_1 \leqq t \leqq t_2)$$

and

$$y_e{}^i(t, 0, 0) \equiv y_e{}^i(t, 0).$$

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[17] By using here the symbol e for the parameter b occurring in the family of the corollary, we retain the notation employed in the lemma.

We choose $l + 1$ sets of arbitrary constants $\bar{u}^j$, $u_h{}^j$, $(j = 1, \cdots, r;$ $h = 1, \cdots, l)$ and define

(65) $$\alpha^j = e\bar{u}^j + b_h u_h{}^j.$$

For all small non-negative values of e and $b_1, \cdots, b_l$ these points are interior to R_2. Substituting these α^j in the functions $\theta(\alpha)$ and $T_s{}^i(\alpha)$ of (32) and (34) respectively, we obtain functions $\theta(e\bar{u} + b_h u_h)$, $T_s{}^i(e\bar{u} + b_h u_h)$ of class C^2 in (e, b). Consequently, the admissible set $(C(e, b), e\bar{u} + b_h u_h)$ gives to the function $J(C, \alpha)$ in (32) and to the functions $y_s{}^i - T_s{}^i(\alpha)$, $(s = 1, 2)$, in the end-conditions (34) the respective values

(66) $$J(C(e, b), e\bar{u} + b_h u_h) \equiv \theta(e\bar{u} + b_h u_h) + I(C(e, b))$$
and
(67) $$y_s{}^i(e, b) - T_s{}^i(e\bar{u} + b_h u_h) \qquad (i = 1, \cdots, n; \, s = 1, 2).$$

These are evidently of class C^2 in e and b_h on $0 \leqq e \leqq \epsilon$, $0 \leqq b_h \leqq \epsilon_h$.

It is easy to verify the equations below for the partial derivatives of the functions in (66) and (67) with respect to e or to b_h at $e = b = 0$. We denote differentiation of the functions θ and $T_s{}^i$ with respect to α^j or α^k by writing the subscript j or k; also we omit writing the argument t belonging to the functions y_0, $\bar{\eta}$, η_h and their derivatives.

(68) $$\frac{\partial}{\partial b_h} J(C(e, b), e\bar{u} + b_h u_h)\big|_{e=b=0} = J_1(\eta_h, u_h)$$
$$\equiv \theta_j(0) u_h{}^j + \int_{t_1}^{t_2} [f_{y^i}(y_0, y_0{}')\eta_h{}^i + f_{r^i}(y_0, y_0{}')\eta_h{}^{i'}] dt + f(y_0(\bar{t}_h), r_h);$$

(69) $$\frac{\partial}{\partial e} J(C(e, b), e\bar{u} + b_h u_h)\big|_{e=b=0} = J_1(\bar{\eta}, \bar{u})$$
$$\equiv \theta_j(0) \bar{u}^j + \int_{t_1}^{t_2} [f_{y^i}(y_0, y_0{}')\bar{\eta}^i + f_{r^i}(y_0, y_0{}')\bar{\eta}^{i'}] dt;$$

(70) $$\frac{\partial^2}{\partial e^2} J(C(e, b), e\bar{u} + b_h u_h)\big|_{e=b=0} = J_2(\bar{\eta}, \bar{u})$$
$$\equiv \theta_{jk}(0) \bar{u}^j \bar{u}^k + \int_{t_1}^{t_2} 2\omega(\bar{\eta}, t, \bar{\eta}') dt$$
$$+ \int_{t_1}^{t_2} [f_{y^i}(y_0, y_0{}')y^i{}_{ee}(t, 0) + f_{r^i}(y_0, y_0{}')y^i{}_{ee}(t, 0)] dt$$

where by definition

$$2\omega(\eta, t, \rho) \equiv f_{y^i y^j}(y_0, y_0{}')\eta^i \eta^j + 2f_{y^i r^j}(y_0, y_0{}')\eta^i \rho^j + f_{r^i r^j}(y_0, y_0{}')\rho^i \rho^j;$$

(71) $$\frac{\partial}{\partial b_h} [y_s{}^i(e, b) - T_s{}^i(e\bar{u} + b_h u_h)]_{e=b=0} = \eta_h{}^i(t_s) - T^i{}_{s,j}(0) u_h{}^j,$$
$$(s = 1, 2)$$

with similar equations holding for the partial derivatives with respect to e, if η_h and u_h are replaced respectively by $\bar{\eta}$ and $\bar{u}$;

$$(72) \quad \frac{\partial^2}{\partial e^2} \left[y_s{}^i(e, b) - T_s{}^i(e\bar{u} + b_h u_h) \right]_{e=b=0} = y^i{}_{ee}(t_s, 0) - T^i{}_{s,jk}(0)\bar{u}^j\bar{u}^k.$$

3. Application of the general theorem to the problem of Bolza. We are now in a position to apply Theorem I. We make the following definitions.

Z is the collection of all admissible sets (C, α). We assume that $z_0 = (C_0, 0)$ is in Z and minimizes the functional $J(C, \alpha)$ on Z, subject to the conditions

$$y_s{}^i = T_s{}^i(\alpha) \qquad\qquad (i = 1, \cdots, n; \; s = 1, 2).$$

V is the aggregate of all vectors $v = (v^0, \cdots, v^{2n})$ in $2n + 1$-dimensional space such that there exists a set (η, u), wherein $\eta = (\eta^1, \cdots, \eta^n)$ is an admissible variation and $u = (u^1, \cdots, u^r)$ is an arbitrary r-tuple of numbers, satisfying the relations

$$(73) \quad \begin{aligned} v^0 &= J_1(\eta, u), \\ v^i &= \eta^i(t_1) - T^i{}_{1,j}(0)u^j \qquad (i = 1, \cdots, n; \; j = 1, \cdots, r), \\ v^{n+i} &= \eta^i(t_2) - T^i{}_{2,j}(0)u^j, \end{aligned}$$

where $J_1(\eta, u)$ is defined formally by the right member of (68) with $h = 1$.

W is the collection of all vectors $w = (w^0, \cdots, w^{2n})$ such that there exists a set $(\bar{\eta}, \bar{u})$, wherein $\bar{\eta} = (\bar{\eta}^1, \cdots, \bar{\eta}^n)$ is an admissible continuous variation and $\bar{u} = (\bar{u}^1, \cdots, \bar{u}^r)$ is an arbitrary r-tuple of numbers, with the properties;

$(74\text{-}i)$ there exists a family of admissible curves of class C^2 in e,

$$y^i = y^i(t, e) \qquad\qquad (t_1 \leqq t \leqq t_2, \; 0 \leqq e \leqq \epsilon)$$

satisfying at $e = 0$ the relations $y_e{}^i(t, 0) = \bar{\eta}^i(t)$,

   ii) $J_1(\bar{\eta}, \bar{u}) \leqq 0$,

   iii) $\bar{\eta}^i(t_s) - T^i{}_{s,j}(0)\bar{u}^j = 0$ $(s = 1, 2)$,

   iv) $w^0 = J_2(\bar{\eta}, \bar{u})$,

$$w^i = y^i{}_{ee}(t_1, 0) - T^i{}_{1,jk}(0)\bar{u}^j\bar{u}^k,$$

$$w^{n+i} = y^i{}_{ee}(t_2, 0) - T^i{}_{2,jk}(0)\bar{u}^j\bar{u}^k \quad (i = 1, \cdots, n; \; j, k = 1, \cdots, r),$$

where $J_1(\bar{\eta}, \bar{u})$ and $J_2(\bar{\eta}, \bar{u})$ are the respective functions formally defined by the right members of (69) and (70).

Now given any vector w in W and any finite collection $v_1, \cdots, v_l$ of

vectors of V, by definition of the sets V and W there exists a collection $(\bar{\eta}, \bar{u})$, (η_1, u_1), $\cdots$, (η_l, u_l) such that the set $(\bar{\eta}, \bar{u})$ has the properties (74, i-iv) and the sets (η_h, u_h), $(h = 1, \cdots, l)$, have the properties (73). By the hypothesis on the set W, there exists a family of admissible curves,

$$C(e): \quad y^i = y^i(t, e) \qquad\qquad (t_1 \leqq t \leqq t_2, \ 0 \leqq e \leqq \epsilon),$$

such that $C(0) \equiv C_0$ and $y_e^i(t, 0) = \bar{\eta}^i(t)$ $(t_1 \leqq t \leqq t_2)$. By the embedding lemma, there exists a family of admissible curves,

$$C(e, b): \quad y^i = y^i(t, e, b_1, \cdots, b_l) \quad (t_1 \leqq t \leqq t_2, \ 0 \leqq e \leqq \epsilon, \ 0 \leqq b_h \leqq \epsilon_h)$$

such that

$$C(e, 0) \equiv C(e), \qquad C(0, 0) \equiv C(0) \equiv C_0,$$

and also

$$y^i{}_{b_h}(t, 0, 0) = \eta_h{}^i(t) \qquad\qquad (t_1 \leqq t \leqq t_2; \ h = 1, \cdots, l)$$

and

$$y_e^i(t, 0, 0) = y_e^i(t, 0).$$

Choose $\alpha^j = e\bar{u}^j + b_h u_h{}^j$ as in (65). Then $z(e, b) = (C(e, b), e\bar{u} + b_h u_h)$ is in Z and $z(0, 0) = (C_0, 0) = z_0$. The functions $J(C(e, b), e\bar{u} + b_h u_h)$ and $y_s^i(e, b) - T_s^i(e\bar{u} + b_h u_h)$ are of class C^2 on $t_1 \leqq t \leqq t_2$, $0 \leqq e \leqq \epsilon$, $0 \leqq b_h \leqq \epsilon_h$. Furthermore by (68), (71) and (73),

$$\frac{\partial}{\partial b_h}(J(C(e, b), e\bar{u} + b_h u_h)|_{e=b=0} = v_h^0 \qquad\qquad (h = 1, \cdots, l),$$

$$\frac{\partial}{\partial b_h}[y_s^i(e, b) - T_s^i(e\bar{u} + b_h u_h)]_{e=b=0} = v^{i+n(s-1)} \quad (i = 1, \cdots, n; \ s = 1, 2)$$

and also by (69), (70), (71), (72) and (74, i-iv),

$$\frac{\partial}{\partial e}J(C(e, b), e\bar{u} + b_h u_h)|_{e=b=0} \leqq 0,$$

$$\frac{\partial}{\partial e}[y_s^i(e, b) - T_s^i(e\bar{u} + b_h u_h)]_{e=b=0} = 0 \qquad\qquad (s = 1, 2),$$

$$\frac{\partial^2}{\partial e^2}J(C(e, b), e\bar{u} + b_h u_h)|_{e=b=0} = w^0,$$

$$\frac{\partial^2}{\partial e^2}[y_s^i(e, b) - T_s^i(e\bar{u} + b_h u_h)]_{e=b=0} = w^{i+n(s-1)} \qquad (s = 1, 2).$$

Consequently the hypotheses of Theorem I are satisfied.

We observe that if $\bar{\eta}(t)$ is a continuous admissible variation and $\bar{u}$ an arbitrary r-tuple such that the set $(\bar{\eta}, \bar{u})$ has the properties, (74, i and iii), then $-\bar{\eta}$ is also continuous and admissible and $(-\bar{\eta}, -\bar{u})$ likewise has these two properties. For by the corollary cited above the curves of the family $y^i(t, e)$ satisfying (74-i) are defined on $-\epsilon \leqq e \leqq +\epsilon$. Consequently under

a mapping of $[0, -\epsilon]$ onto $[0, \epsilon]$ such that $+ e$ corresponds to $- e$, we obtain a family $y^{*i}(t, e)$, $(0 \leqq e \leqq \epsilon)$, for which the equations

$$y^{*i}_{e}(t, 0) = -\bar{\eta}^i(t) \qquad\qquad (t_1 \leqq t \leqq t_2)$$

and also

$$y^{*i}_{ee}(t, 0) = y^i_{ee}(t, 0)$$

hold. Thus the functions in the right members of (74-iv) are not changed if $(\bar{\eta}, \bar{u})$ is replaced by $(-\bar{\eta}, -\bar{u})$. Hence it is immaterial whether or not a set $(\bar{\eta}, \bar{u})$ which possesses all the properties needed for the definition of a vector of W, except possibly (74-ii), satisfies that condition also. For, if not, by a change in signs we obtain a set which does have all the properties (74) and hence defines a vector w. By the remarks above, the right-hand members of (74-iv) are identical for these two sets.

Hence we have the theorem.

THEOREM II. *Let the set* $(C_0, 0)$ *consisting of the curve*

$$C_0: \quad y^i = y_0^i(t) \qquad\qquad (t_1 \leqq t \leqq t_2; \; i = 1, \cdots, n)$$

and the r-tuple

$$\alpha = (\alpha^1, \cdots, \alpha^r) = (0, \cdots, 0)$$

minimize the functional

$$J(C, \alpha) \equiv \theta(\alpha) + \int_{t_1}^{t_2} f(y(t), y'(t)) \, dt$$

on the class of admissible sets (C, α) *satisfying the differential equations*

$$\phi^\beta(y(t), y'(t)) = 0 \qquad\qquad (\beta = 1, \cdots, m < n - 1)$$

and the end-conditions

$$y_s^i = T_s^i(\alpha) \qquad\qquad (s = 1, 2).$$

Let $\bar{\eta}(t) = (\bar{\eta}^1(t), \cdots, \bar{\eta}^n(t))$ *be a continuous admissible variation and* $\bar{u} = (\bar{u}^1, \cdots, \bar{u}^r)$ *a set of numbers such that*

$$\bar{\eta}^i(t_s) - T^i_{s,j}(0)\bar{u}^j = 0$$

By the corollary to the embedding lemma there exists a family of admissible curves

$$y^i = y^i(t, e) \qquad\qquad (t_1 \leqq t \leqq t_2, \; -\epsilon \leqq e \leqq +\epsilon)$$

satisfying on $[t_1, t_2]$ *the relations*

$$y^i(t, 0) = y_0{}^i(t),$$
$$y_e{}^i(t, 0) = \bar{\eta}^i(t),$$
$$\phi^\beta(y(t, e), y'(t, e)) \equiv 0.$$

Let $y^i(t, e)$ be such a family.

Then there exist numbers $l_0 \geqq 0, l_1, \cdots, l_{2n}$, not all zero, such that for every admissible variation $\eta(t)$ and every set of numbers $u = (u^1, \cdots, u^r)$ it is true that

1. $l_0 J_1(\eta, u) + l_i[\eta^i(t_1) - T^i{}_{1,j}(0)u^j] + l_{n+i}[\eta^i(t_2) - T^i{}_{2,j}(0)u^j] \geqq 0$

and also

2. $l_0 J_2(\bar{\eta}, \bar{u}) + l_i[y^i{}_{ee}(t_1, 0) - T^i{}_{1,jk}(0)\bar{u}^j\bar{u}^k]$
    $\qquad + l_{n+i}[y^i{}_{ee}(t_2, 0) - T^i{}_{2,jk}(0)\bar{u}^j\bar{u}^k] \geqq 0$
    $$(i = 1, \cdots, n; \; j, k = 1, \cdots, r).$$

4. Necessary conditions for a minimum. By the usual method [18] we can prove that, with the exception of the Jacobi condition, the necessary conditions for a minimum follow as a consequence of the first inequality in the conclusion of Theorem II. This we shall do briefly.

First we state without proof a lemma due to Bliss.[19]

LEMMA. *If $\lambda_0, c_1, \cdots, c_n$ are arbitrary constants, there exists a uniquely determined set of functions $\lambda_1(t), \cdots, \lambda_n(t), (t_1 \leqq t \leqq t_2)$, such that if we define*

(75) $F(y, r, t, \lambda) \equiv \lambda_0 f(y, r) + \lambda_\beta \phi^\beta(y, r) + \lambda_\gamma \phi^\gamma(t, r),$

the equations

(76) $F_{r^i}(y_0, y_0', t, \lambda) = \int_{t_1}^{t} F_{y^i}(y_0, y_0', t, \lambda)dt + c_i$

are satisfied at every point of the minimizing arc C_0.

Now the first inequality of Theorem II holds for every set (η, u) in which $\eta(t)$ is an admissible variation and u is an arbitrary r-tuple of numbers. In particular, it holds for every (η, u) wherein $\eta(t)$ is a continuous admissible variation. But then $(-\eta, -u)$ is also a set such that $-\eta$ is a continuous admissible variation. This implies that

(77) $l_0[\theta_j(0)u^j] + \int_{t_1}^{t_2} [f_{y^i}(y_0(t), y_0'(t))\eta^i(t) + f_{r^i}(y_0(t), y_0'(t))\eta^{i'}(t)]dt$
    $\qquad + l_i[\eta^i(t_1) - T^i{}_{1,j}(0)u^j] + l_{n+i}[\eta^i(t_2) - T^i{}_{2,j}(0)u^j] = 0.$

[18] G. A. Bliss, $(1, 2)$; E. J. McShane, (6); F. G. Myers, (10).
[19] *Loc. cit.*, $(2, \text{p. } 683)$.

Recall that every admissible variation $\eta(t)$ defines uniquely a set of functions $\zeta^\gamma(t)$ by the transformation (38). By Bliss' lemma, every set of constants $\lambda_0, c_1, \cdots, c_n$ determines a set of functions $\lambda_i(t)$ such that for the function $F(y, r, t, \lambda)$ defined by (75) the equations (76) hold. We choose $\lambda_0 = l_0 \geqq 0$. For the moment we reserve the choice of the c_i. Making use of the sets $\lambda_i(t)$ and $\zeta^\gamma(t)$ we add the identity

$$\int_{t_1}^{t_2} [\lambda_\beta(t)\Phi^\beta(\eta, t, \eta') + \lambda_\gamma(t)(\Phi^\gamma(\eta, t, \eta') - \zeta^\gamma(t))]dt \equiv 0$$

to the equation (77) getting with the help of (75) the equation

$$(78) \quad \lambda_0\theta_j(0)u^j + \int_{t_1}^{t_2}[F_{y^i}(y_0, y_0', t, \lambda)\eta^i(t) + F_{r^i}(y_0, y_0', t, \lambda)\eta^{i\prime}(t)]dt$$
$$- \int_{t_1}^{t_2}\lambda_\gamma(t)\zeta^\gamma(t)dt + l_i[\eta^i(t_1) - T^i_{1,j}(0)u^j] + l_{n+i}[\eta^i(t_2) - T^i_{2,j}(0)u^j] = 0.$$

The usual integration by parts applied to the first integral in (78) yields

$$\eta^i(t)\int_{t_1}^{t} F_{y^i}(y_0, y_0', t, \lambda)dt \bigg|_{t_1}^{t_2}$$
$$+ \int_{t_1}^{t_2}\eta^{i\prime}(t)[F_{r^i}(y_0, y_0', t, \lambda) - \int_{t_1}^{t} F_{y^i}(y_0, y_0', t, \lambda)dt]dt.$$

Substituting this expression in (78) and making use of (76), we obtain, after rearranging terms, the equation

$$(79) \quad \eta^i(t_2)\Big[\int_{t_1}^{t_2}F_{y^i}(y_0, y_0', t, \lambda)dt + c_i + l_{n+i}\Big] + \eta^i(t_1)(l_i - c_i)$$
$$+ u^j[\lambda_0\theta_j(0) - l_iT^i_{1,j}(0) - l_{n+i}T^i_{2,j}(0)] - \int_{t_1}^{t_2}\lambda_\gamma(t)\zeta^\gamma(t)dt = 0.$$

Now we choose once and for all $c_i = l_i$, where the numbers l_i are the multipliers given by Theorem II. Hence by Bliss' lemma, corresponding to the set of constants

$$(80) \qquad\qquad \lambda_0 = l_0 \geqq 0, \qquad c_i = l_i \qquad\qquad (i = 1, \cdots, n)$$

there exists a unique set of functions $\lambda_i(t)$ for which (76) and consequently (79) hold. For the set of constants (80), it follows from Theorem II and (77) that the equation (79) must be satisfied for every set of constants u^j and every admissible continuous variation $\eta(t)$; that is, for every choice of the functions $\zeta^\gamma(t)$ and of the numbers $\eta^i(t_2)$ since by Lemma 2 for every such choice the admissible variation $\eta(t)$ is uniquely determined on $[t_1, t_2]$. Hence we have still at our disposal the choice of the functions $\zeta^\gamma(t)$ and the constants $\eta^i(t_2)$ and u^j.

We choose these as follows,

$$\zeta^\gamma(t) = \lambda_\gamma(t),$$

$$\eta^i(t_2) = -\left[\int_{t_1}^{t_2} F_{y^i}(y_0, y_0', t, \lambda)dt + l_i + l_{n+i}\right],$$

$$u^j = -\left[\lambda_0\theta_j(0) - l_i T^i_{1,j}(0) - l_{n+i} T^i_{2,j}(0)\right].$$

Upon substitution of these values in (79) we obtain the equation

$$-\left[\int_{t_1}^{t_2} F_{y^i}(y_0, y_0', t, \lambda)dt + l_i + l_{n+i}\right]^2$$
$$-\left[\lambda_0\theta_j(0) - l_i T^i_{1,j}(0) - l_{n+i} T^i_{2,j}(0)\right]^2 - \int_{t_1}^{t_2}(\lambda_\gamma(t))^2 dt = 0$$

from which it follows that the equations

$$(81) \qquad \int_{t_1}^{t_2}(\lambda_\gamma(t))^2 dt = 0,$$

$$(82) \qquad \int_{t_1}^{t_2} F_{y^i}(y_0, y_0', t, \lambda)dt + l_i + l_{n+i} = 0$$

and

$$(83) \qquad \lambda_0\theta_j(0) - l_i T^i_{1,j}(0) - l_{n+i} T^i_{2,j}(0) = 0$$

must be satisfied. Since the $\lambda_\gamma(t)$ are of class D^0, continuous between corners of C_0, the equation (81) yields

$$\lambda_\gamma(t) \equiv 0 \qquad\qquad (\gamma = m + 1, \cdots, n).$$

The only reason for the presence of the argument t in the function $F(y, r, t, \lambda)$ is the term $\lambda_\gamma\phi^\gamma(t, r)$ in the definition. Now this term is missing, so hereafter F will be written $F(y, r, \lambda)$ instead. Making use of (76) and (80) we find that the equations (82) yield

$$(84) \qquad l_{n+i} = -F_{r^i}(y_0(t_2), y_0'(t_2), \lambda(t_2))$$

and

$$(85) \qquad l_i = c_i = F_{r^i}(y_0(t_1), y_0'(t_1), \lambda(t_1)).$$

Consequently the equations (83) have the form

$$(86) \quad \lambda_0\theta_j(0) - F_{r^i}(y_0(t_1), y_0'(t_1), \lambda(t_1)) T^i_{1,j}(0)$$
$$+ F_{r^i}(y_0(t_2), y_0'(t_2), \lambda(t_2)) T^i_{2,j}(0) = 0.$$

These are the transversality conditions.

Thus the first inequality in Theorem II implies that there exist multipliers $\lambda_0 \geqq 0$, $\lambda_1(t), \cdots, \lambda_m(t)$ continuous between corners of C_0 such that for the function $F(y, r, \lambda)$ it is true that

I. *the Dubois-Reymond relations* (76) *hold on* $t_1 \leqq t \leqq t_2$; *between corners of C_0 the analogues of the Euler-Lagrange equations*

$$\frac{d}{dt} F_{r^i}(y_0(t), y_0'(t), \lambda(t)) = F_{y^i}(y_0(t), y_0'(t), \lambda(t))$$

hold and

Ia. *the transversality conditions* (86) *are satisfied at t_1 and t_2.*

The multipliers λ_0, $\lambda_\beta(t)$, $(\beta = 1, \cdots, m)$, *do not all vanish at any one t.* To prove this we note that if $\lambda_0 = 0$, the Euler equations reduce to

$$\frac{d}{dt}(\lambda_\beta(t)\phi^\beta_{r^i}(y_0(t), y_0'(t))) = \lambda_\beta(t)\phi^\beta_{y^i}(y_0(t), y_0'(t))$$

or

$$\lambda_\beta'(t)\phi^\beta_{r^i}(y_0, y_0') = -\lambda_\beta(t)\frac{d}{dt}\phi^\beta_{r^i}(y_0, y_0') + \lambda_\beta(t)\phi^\beta_{y^i}(y_0, y_0').$$

Because of the non-singularity of the matrix $\| \phi^\beta_{r^i}(y_0, y_0') \|$ we can pick out m linearly independent equations, which we can solve for $\lambda_\beta'(t)$ as linear homogeneous functions of $\lambda_\beta(t)$. It follows from the properties of differential equations of this type that if the $\lambda_\beta(t)$ are all zero for any one value of t, then $\lambda_\beta(t) \equiv 0$, $(t_1 \leqq t \leqq t_2, \beta = 1, \cdots, m)$. Consequently $F(y_0(t), y_0'(t), \lambda(t)) \equiv 0$ and by (84) and (85) the multipliers $l_1, \cdots, l_{2n}$ together with $l_0 = \lambda_0$ would all be zero. This contradiction of Theorem II proves our statement.

The *Weierstrass-Erdmann corner condition,*

$$F_{r^i}(y_0(t-), y_0'(t-), \lambda(t-)) = F_{r^i}(y_0(t+), y_0'(t+), \lambda(t+)),$$

at any point t defining a corner of C_0, is an immediate consequence of (76).

We come next to the analogue of the Weierstrass condition:

II. *For all t in the interval $[t_1, t_2]$ and all r such that $(y_0(t), r)$ is admissible, the inequality*

(87) $E(y_0(t), y_0'(t), r, \lambda(t)) \equiv F(y_0(t), r, \lambda(t)) - r^i F_{r^i}(y_0(t), y_0'(t), \lambda(t)) \geqq 0$

holds.

Let t_0 be any point in $[t_1, t_2]$, and let $(y_0(t_0), r_0)$ be admissible. Without loss of generality, we may suppose that r_0 is a unit vector. We choose a set of constants $(u^1, \cdots, u^r) \equiv (0, \cdots, 0)$ and also a set of functions $\zeta^\gamma(t)$, $(\gamma = m + 1, \cdots, n)$, identically zero on $[t_1, t_2]$. By applying Lemma 2 first to $[t_1, t_0]$ and then to $[t_0, t_2]$, we obtain an admissible variation $\eta(t)$ such that

(88) $$\eta^i(t_0 -) = 0 \quad \text{and} \quad \eta^i(t_0 +) = r_0{}^i$$

while the equations

(89) $$\Phi^\beta(\eta, t, \eta') = 0, \qquad \Phi^\gamma(\eta, t, \eta') = 0$$

hold. Referring to equations (68), (71), (84), and (85), we see that the first inequality of Theorem II now has the form

$$(90) \quad \lambda_0 \int_{t_1}^{t_2} (f_{y^i}(y_0(t), y_0{}'(t))\eta^i(t) + f_{r^i}(y_0(t), y_0{}'(t))\eta^{i'}(t)) dt$$
$$+ \lambda_0 f(y_0(t_0), r_0) - F_{r^i}(y_0(t_2), y_0{}'(t_2), \lambda(t_2))\eta^i(t_2)$$
$$+ F_{r^i}(y_0(t_1), y_0{}'(t_1), \lambda(t_1))\eta^i(t_1) \geqq 0.$$

Since the functions $\zeta^\gamma(t)$ vanish identically, equations (89) are linear homogeneous differential equations for $\eta^i(t)$, and since the determinant of the coefficients of $\eta^{i'}(t)$ is non-singular, we can solve for these derivatives. On the interval $[t_1, t_0]$, the functions $\eta^i(t)$ satisfy these equations and have the final values $\eta^i(t_0 -) = 0$; hence we obtain the identities

(91) $$\eta^i(t) \equiv 0. \qquad\qquad (t_1 \leqq t \leqq t_0).$$

If we add the functions $\lambda_\beta(t)\Phi^\beta(\eta, t, \eta')$, which are identically zero by (89), to the integrand in (90) and recall (91), this inequality becomes

$$(92) \quad \int_{t_0}^{t_2} [F_{y^i}(y_0(t), y_0{}'(t), \lambda(t))\eta^i(t) + F_{r^i}(y_0(t), y_0{}'(t), \lambda(t))\eta^{i'}(t)] dt$$
$$- F_{r^i}(y_0(t_2), y_0{}'(t_2), \lambda(t_2))\eta^i(t_2) + F_{r^i}(y_0(t_1), y_0{}'(t_1), \lambda(t_1))\eta^i(t_1)$$
$$+ \lambda_0 f(y_0(t_0), r_0) \geqq 0.$$

Because of (91), the usual integration by parts applied to the first term in the above integrand transforms the integral into

$$\eta^i(t) \int_{t_1}^{t} F_{y^i}(y_0, y_0{}', \lambda) dt \Big|_{t_0}^{t_2} + \int_{t_0}^{t_2} \eta^{i'}(t) [F_{r^i}(y_0, y_0{}', \lambda) - \int_{t_1}^{t} F_{y^i}(y_0, y_0{}', \lambda) dt] dt.$$

Substituting this expression in (92), we obtain with the help of (76), (85), (88), and (91) the inequality

$$\lambda_0 f(y_0(t_0), r_0) - r_0{}^i F_{r^i}(y_0(t_0), y_0{}'(t_0), \lambda(t_0)) \geqq 0.$$

Upon addition of $\lambda_\beta(t_0)\phi^\beta(y_0(t_0), r_0)$, which is identically zero because of the admissibility of $(y_0(t_0), r_0)$, the last inequality becomes

$$F(y_0(t_0), r_0, \lambda(t_0)) - r_0{}^i F_{r^i}(y_0(t_0), y_0{}'(t_0), \lambda(t_0)) \geqq 0.$$

If t_0 defines a corner of C_0, then in the above proof we understand $y_0{}'(t_0)$ to be the right derivative. By simple continuity considerations the preceding

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inequality is valid at corners if we interpret the y_0' as the left derivative; it will hold also at the end-points t_1, t_2. Thus the Weierstrass condition holds for all t in $[t_1, t_2]$.

By the usual methods [20] we can show that as a consequence of the Weierstrass condition we obtain the analogue of the Clebsch condition:

III. *For all t in $[t_1, t_2]$ and all numbers $\pi^1, \cdots, \pi^n$ such that*

$$\phi^\beta_{r^i}(y_0(t), y_0'(t))\pi^i = 0 \qquad\qquad (\beta = 1, \cdots, m)$$

the inequality

$$F_{r^i r^j}(y_0(t), y_0'(t), \lambda(t))\pi^i\pi^j \geqq 0$$

is satisfied.

Likewise from conditions II and III and the Weierstrass-Erdmann corner condition we can get the Dresden corner condition: [21]

If C_0 satisfies I and II with multipliers $\lambda_0 \geqq 0, \lambda_1(t), \cdots, \lambda_m(t)$ and t_0 defines a corner of C_0, then the inequality

$$\Omega(y_0(t_0)), (y_0'(t_0 -), y_0'(t_0 +), \lambda(t))$$
$$\equiv y_0^{i'}(t_0 -) F_{y^i}(y_0(t_0), y_0'(t_0 +), \lambda(t_0 +))$$
$$- y_0^{i'}(t_0 +) F_{y^i}(y_0(t_0), y_0'(t_0 -), \lambda(t_0 -)) \leqq 0$$

holds.

Thus we have verified the statement made at the beginning of this section. In so far as the first inequality in the conclusions of Theorems I and II and its consequences are concerned, this paper offers nothing new; these results having been previously obtained by McShane.[22]

Now let the set $(\bar\eta, \bar u)$ be such that $\bar\eta(t)$ is an admissible continuous variation and $\bar u = (\bar u^1, \cdots, \bar u^n)$ is an r-tuple of numbers satisfying the equations

(93) $$\bar\eta^i(t_s) - T^i_{s,j}(0)\bar u^j = 0 \qquad\qquad (s = 1, 2).$$

For this set $(\bar\eta, \bar u)$ conclusion 1 of Theorem II holds with the equality sign, as we have shown; furthermore the second inequality of that theorem is satisfied. The foregoing argument of this section is valid. In particular, the set of numbers (80) determines uniquely a set of multipliers $\lambda_0 \geqq 0$, $\lambda_1(t), \cdots, \lambda_m(t)$ with the properties proved above for which the necessary conditions I, Ia, II, and III all hold along C_0 for the function $F(y, r, \lambda)$.

[20] See e. g., G. A. Bliss, (2, p. 718).

[21] F. G. Myers, (10).

[22] E. J. McShane, (5, 6).

It is desirable now to express the second inequality in a more familiar form. With the help of (80), (84), and (85), this inequality becomes

$$(94) \quad \lambda_0 J_2(\bar{\eta}, \bar{u})$$
$$- F_{r^i}(y_0(t_s), y_0'(t_s), \lambda(t_s))(y^i_{ee}(t_s, 0) - T^i_{s,hk}(0)\bar{u}^h\bar{u}^k)|^2 \geqq 0$$
$$(h, k = 1, \cdots, r; s = 1, 2)$$

where $J_2(\bar{\eta}, \bar{u})$ is the expression in (70). Differentiating the identities

$$\phi^\beta(y(t, e), y'(t, e)) \equiv 0 \qquad (\beta = 1, \cdots, m)$$

twice with respect to e, we obtain at $e = 0$ the equations

$$(95) \quad \phi^\beta_{y^i y^j}\bar{\eta}^i\bar{\eta}^j + 2\phi^\beta_{y^i r^j}\bar{\eta}^i\bar{\eta}^{j'} + \phi^\beta_{r^i r^j}\bar{\eta}^{i'}\bar{\eta}^{j'} + \phi^\beta_{y^i}y^i_{ee}(t, 0) + \phi^\beta_{r^i}y^{i'}_{ee}(t, 0) = 0$$

where the arguments in the derivatives of ϕ^β are the functions $y_0^i(t)$ and $y_0^{i'}(t)$ belonging to C_0. If we multiply the equations (95) respectively by $\lambda_\beta(t)$, integrate from t_1 to t_2 and add the sum of the integrals to (94), we do not change the value of that expression. Thus if we define

$$2\omega(\eta, t, \rho, \lambda) \equiv F_{y^i y^j}(y_0, y_0', \lambda)\eta^i\eta^j + 2F_{y^i r^j}(y_0, y_0', \lambda)\eta^i\rho^j + F_{r^i r^j}(y_0, y_0', \lambda)\rho^i\rho^j,$$

we obtain the inequality (94) in the form

$$(96) \quad J_2(\bar{\eta}, \bar{u}, \lambda) = \lambda_0\theta_{hk}(0)\bar{u}^h\bar{u}^k + \int_{t_1}^{t_2} 2\omega(\bar{\eta}, t, \bar{\eta}', \lambda)\,dt$$
$$+ \int_{t_1}^{t_2} [F_{y^i}(y_0, y_0', \lambda)y^i_{ee}(t, 0) + F_{r^i}(y_0, y_0', \lambda)y^{i'}_{ee}(t, 0)]\,dt$$
$$- F_{r^i}(y_0(t_s), y_0'(t_s), \lambda(t_s))(y^i_{ee}(t_s, 0) - T^i_{s,hk}(0)\bar{u}^h\bar{u}^k)|_1^2 \geqq 0.$$

Now we integrate by parts the second term in the second integral of (96), applying the process from corner to corner of C_0, and then make use of the Euler equations which hold for the function $F(y_0, y_0', \lambda)$ between corners of C_0. This transforms that integral into the expression

$$y^i_{ee}(t, 0)F_{r^i}(y_0(t), y_0'(t), \lambda(t))\Big|_{t_1}^{t_2}.$$

Upon substituting this in (96) and collecting terms, we find that all terms containing the derivative y^i_{ee} vanish. Hence the inequality (96) reduces to

$$(97) \qquad J_2(\bar{\eta}, \bar{u}, \lambda) = b_{hk}\bar{u}^h\bar{u}^k + \int_{t_1}^{t_2} 2\omega(\bar{\eta}, t, \bar{\eta}', \lambda)\,dt \geqq 0$$

where b_{hk} are the constants defined by

$$b_{hk} \equiv \lambda_0\theta_{hk}(0) + [F_{r^i}(y_0(t_s), y_0'(t_s), \lambda(t_s))T^i_{s,hk}(0)]\Big|_1^2.$$

Collecting the various statements in section **4**, we have the following theorem:

THEOREM IV. *Let the set* $(C_0, 0)$ *consisting of the curve*

$$C_0: \quad y^i = y_0{}^i(t) \qquad\qquad (t_1 \leqq t \leqq t_2; \; i = 1, \cdots, n)$$

and the numbers

$$\alpha = (\alpha^1, \cdots, \alpha^r) = (0, \cdots, 0)$$

minimize the functional

$$J(C, \alpha) \equiv \theta(\alpha) + \int_{t_1}^{t_2} f(y(t), y'(t)) dt$$

on the class of admissible sets (C, α) *satisfying the differential equations*

$$\phi^\beta(y(t), y'(t)) = 0 \qquad\qquad (\beta = 1, \cdots, m < n-1)$$

and the end-conditions

$$y^i(t_s) - T_s{}^i(\alpha) = 0 \qquad\qquad (i = 1, \cdots, n; \; s = 1, 2).$$

Then if the set $(\bar{\eta}, \bar{u})$ *is such that* $\bar{\eta}(t) = (\bar{\eta}^1(t), \cdots, \bar{\eta}^n(t))$ *is an admissible variation and is continuous on* $[t_1, t_2]$ *and* $\bar{u} = (\bar{u}^1, \cdots, \bar{u}^r)$ *are numbers satisfying*

$$\bar{\eta}^i(t_s) = T^i{}_{s,j}(0)\bar{u}^j,$$

it is true that there exists a non-negative constant λ_0 *and a set of functions* $\lambda_1(t), \cdots, \lambda_m(t)$ *such that for the function*

$$F(y, y', \lambda) \equiv \lambda_0 f + \lambda_1 \phi^1 + \cdots + \lambda_m \phi^m$$

the following statements hold

I. (DuBois-Reymond relations)

There are constants $c_1, \cdots, c_n$ such that the equations

$$F_{r^i}(y_0(t), y_0'(t), \lambda(t)) = \int_{t_1}^{t} F_{y^i}(y_0(t), y_0'(t), \lambda(t)) dt + c_i$$

hold on the entire interval $[t_1, t_2]$.

Ia. (Transversality conditions)

At the end-points of the interval $[t_1, t_2]$ the conditions

$$\lambda_0 \theta_j(0) + \left[F_{r^i}(y_0(t_s), y_0'(t_s), \lambda(t_s)) T^i{}_{s,j}(0) \right]_1^2 = 0$$

are satisfied for each $j = 1, \cdots, r$.

II. (Weierstrass condition)

For all t in the interval $[t_1, t_2]$ and all r such that $(y_0(t), r)$ is admissible, the inequality

$$E(y_0(t), y_0'(t), r, \lambda(t)) \geqq 0$$

is satisfied.

III. (Clebsch condition)

For all t in $[t_1, t_2]$ and all sets of numbers $\pi^1, \cdots, \pi^n$ satisfying the equations

$$\phi^\beta{}_{r^i}(y_0(t), y_0'(t))\pi^i = 0 \qquad\qquad (\beta = 1, \cdots, m)$$

the inequality

$$F_{r^i r^j}(y_0(t), y_0'(t), \lambda(t))\pi^i \pi^j \geqq 0$$

holds.

IV. *The function*

$$J_2(\bar{\eta}, \bar{u}, \lambda) \equiv b_{hk}\bar{u}^h \bar{u}^k + \int_{t_1}^{t_2} 2\omega(\bar{\eta}, t, \overline{\eta'}, \lambda)\,dt$$

in which

$$2\omega(\bar{\eta}, t, \overline{\eta'}, \lambda) = F_{y^i y^j}(y_0(t), y_0'(t), \lambda(t))\bar{\eta}^i \bar{\eta}^j$$
$$+ 2F_{y^i r^j}(y_0(t), y_0'(t), \lambda(t))\bar{\eta}^i \bar{\eta}^{j'} + F_{r^i r^j}(y_0(t), y_0'(t), \lambda(t))\bar{\eta}^{i'} \bar{\eta}^{j'}$$

and

$$b_{hk} = \lambda_0 \theta_{hk}(0) + \left[F_{r^i}(y_0(t_s), y_0'(t_s), \lambda(t_s))T^i{}_{s,hk}(0) \right]_{s=1}^{s=2}$$

satisfies the inequality

$$J_2(\bar{\eta}, \bar{u}, \lambda) \geqq 0.$$

Moreover the constant λ_0 and the functions $\lambda_\beta(t)$ can not all vanish at any one point of the interval $[t_1, t_2]$, and the $\lambda_\beta(t)$ are continuous except possibly at values of t defining corners of C_0.

COROLLARY 1. (Euler-Lagrange equations)

Between corners of C_0, the equations

$$\frac{d}{dt} F_{r^i}(y_0(t), y_0'(t), \lambda(t)) = F_{y^i}(y_0(t), y_0'(t), \lambda(t))$$

hold.

COROLLARY 2.   (Weierstrass-Erdmann corner conditions)

At each corner of C_0, the functions F_{r^i} have well defined right and left limits which are equal; that is, if t_0 defines a corner of C_0, then

$$F_{r^i}(y_0(t_0), y_0'(t_0 -), \lambda(t_0 -)) = F_{r^i}(y_0(t_0), y_0'(t_0 +), \lambda(t_0 +)).$$

COROLLARY 3.   (Dresden corner condition)

If t_0 defines a corner of C_0, then the inequality

$$\Omega(y_0(t_0), y_0'(t_0 -), y_0'(t_0 +), \lambda(t))$$
$$\equiv y_0^{i'}(t_0 -) F_{y^i}(y_0(t_0), y_0'(t_0 +), \lambda(t_0 +))$$
$$- y_0^{i'}(t_0 +) F_{y^i}(y_0(t_0), y_0'(t_0 -), \lambda(t_0 -)) \leqq 0$$

holds.

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ASYMPTOTIC DISTRIBUTION OF ZEROS FOR CERTAIN EXPONENTIAL SUMS.*

By H. L. TURRITTIN.

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1. Introduction. L. A. Mac Coll [1][1] has given asymptotic approximations for the number and location of the zeros of a function

$$(1) \qquad E(z) = \sum_{j=1}^{J} P_j \exp. q_j(z)$$

when each q represents a polynomial and each P a constant. R. E. Langer [2] has studied the zero distribution when each P represents an analytic function behaving as a power of z, $|z|$ large, and each q is a polynomial of the first degree. In this paper functions are treated in which both the P's and q's are polynomials of a complex variable z.

The behavior of the absolute values of the individual terms of E are studied as $z \to \infty$ along certain ξ-curves. Polygonal diagrams are used in sorting the terms according to size. This leads to a decomposition of the entire z-plane into a finite number of subregions: the W-regions (without zeros), and the Z-regions (with zeros).

All adjoining Z-regions are lumped together into single larger O-bands. The number of zeros in each O is computed by estimating the variation in amplitude of E as z travels once counter-clockwise around carefully chosen closed contours. Only the zeros exterior to a sufficiently large, fixed circle $|z| = \rho_0$, called the C_0-circle, are counted; those within C_0 are ignored.

The O-bands begin at this C_0-circle, are curvilinear in form, extend out to infinity in a radial fashion, and cluster about a finite number of rays running out from the origin. They fall into three categories: bands bounded (A) by two curves each asymptotic respectively to a different line, both lines parallel to the same ray; (B) by two curves receding from a ray in such a manner that the band approaches constant width as it runs out into the remote portion of the z-plane; and (C) by two curves asymptotically approaching a line which runs parallel to a ray.

The number of zeros, $\mathcal{N}(\rho)$, in a particular O-band between C_0 and a larger concentric circle of radius ρ is given by a formula $\mathcal{N}(\rho) = k\rho^n\{1 + O(1/\rho)\}$.

[1] References are listed at the end of this paper.

Mac Coll [1] and Langer [2] draw the same conclusion, but this approxima-
tion can be improved if ρ is restricted to a suitable set of values $\rho_0, \rho_1, \rho_2, \cdots$
forming a relatively dense set.[2] In fact there exists a polynomial $\mathcal{P}(\rho)$ corre-
sponding to each O-band such that $\mathcal{N}(\rho) = \mathcal{P}(\rho) + O(1)$, $\rho = \rho_0, \rho_1, \cdots$.
This improved estimate is based on our more precise method of locating zero-
free regions and on Theorem II which is an extension of H. Bohr's [3] work
on Almost Periodic Plane Motion.   •

In **2** complex valued F^* functions of two real variables ρ and σ are con-
sidered.   Lower bounds on $|F^*|$ are given in Corollary I.   Then the varia-
tion in amplitude of $F^*(\rho, 0)$ is evaluated in **3**; thus preparing the way for
the detailed study of the given $E(z)$ in the remaining sections **4-11**.

2.   E*-functions and lower bounds.

By definition $F^*_N(\rho, \sigma)$ is an E^*-
function of order N if

$$F^*_N(\rho, \sigma) = \sum_{j=1}^{J} \exp.\{d_j - 2\pi N b_{jN}\sigma + 2\pi i[\mathcal{P}_j(\rho) + b_{j0}]\}, \qquad J > 1,$$

where the d's and b's are real constants, ρ and σ are independent real variables,
$i = \sqrt{-1}$, no two polynomials

$$\mathcal{P}_j(\rho) = b_{jN}\rho^N + b_{j,N-1}\rho^{N-1} + \cdots + b_{j1}\rho$$

are identical, and at least two b_{jN}'s are distinct.

The terms of such an E^*-function of the M-th order can be regrouped
and the function written in the form

$$(2) \qquad F^*_M(\rho, \sigma) = \sum_{k=1}^{K} \Phi_{M-1,k}(\rho) \exp \{2\pi b_{kM}(-M\sigma + i\rho^M)\}, \qquad K > 1,$$

where

$$(3) \qquad\qquad\qquad b_{1M} < b_{2M} < \cdots < b_{KM}$$

and each

$$\Phi_{M-1,k}(\rho) = \sum_{j=1}^{J_k} \exp \{d_j + 2\pi i[\mathcal{P}_j(\rho) + b_{j0}]\}.$$

The d's and b's are real constants and no two of the polynomials

$$\mathcal{P}_j(\rho) = b_{j,M-1}\rho^{M-1} + \cdots + b_{j1}\rho$$

in any particular Φ are identical.

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[2] A given set S of real numbers is "relatively dense" if there exists a constant L
such that at least one member of S is located in every interval of length L on the
positive real axis.

THEOREM I. *If F^*_M is any function of a given finite set S of E^*-functions of orders not exceeding N and if the η's are defined by writing*

$$F^*_M(\rho, \sigma) = \{1 + \eta_{kM}(\rho, \sigma)\}\Phi_{M-1,k}(\rho) \exp\{2\pi b_{kM}(-M\sigma + i\rho^M)\},$$

then there exist for this given S two positive constants Δ and $w < 1$ and a relatively dense set of real numbers $\rho_0, \rho_1, \rho_2, \cdots$ such that

$$(4) \quad |\Phi_{M-1,k}(\rho_j)| > w, \quad (k = 1, \cdots, K); \quad |F^*_M(\rho_j, \sigma)| > w, \quad |\sigma| \leqq \Delta;$$
$$|\eta_{1M}(\rho_j, \sigma)| < .5 - .25^N, \ \sigma > \Delta; \ \text{and} \ |\eta_{KM}(\rho_j, \sigma)| < .5 - .25^N, \ |\sigma| < -\Delta$$
$$(j = 0, 1, 2, \cdots).$$

Proof. Assume that Theorem I is true when $N = 1$. As a first step toward completing the induction when $N > 1$, replace M by N in (2) and factor out the first term in each $\Phi_{N-1,k}$. Thus

$$(5) \qquad \Phi_{N-1,k}(\rho) = \Psi_k(\rho) \exp\{d_1 + 2\pi i[\mathcal{P}_1(\rho) + b_{10}]\}$$

with

$$\Psi_k(\rho) = \sum_{j=1}^{J_k} \exp\{D_j + 2\pi i(B_{jH}\rho^H + \cdots + B_{j1}\rho + B_{j0})\}$$

where $H \leqq N - 1$ and the B's and D's are real constants. If $J_k = 1$, $\Psi_k(\rho) \equiv 1$. If $J_k > 1$, at least two of the B_{jH}'s in a particular Ψ_k are different and no two of the polynomials

$$B_{jH}\rho^H + \cdots + B_{j1}\rho$$

in any particular Ψ_k are identical. All the Ψ's not identically equal to unity are E^*-functions of orders less than N with $\sigma = 0$.

By hypothesis, Theorem I applies to the set S_1, made up of the Ψ_k's and all the F^*_H's of S of orders $H < N$; i. e. there exist constants Δ_1, $w_1 < 1$, and ρ_1 such that $|\Psi_k(\rho_1)| > w_1$,

$$(6) \quad \begin{array}{lll} |\Phi_{H-1,k}(\rho_1)| > w_1; & |F^*_H(\rho_1, \sigma)| > w_1, & |\sigma| \leqq \Delta_1; \\ |\eta_{1H}(\rho_1, \sigma)| < .5 - .25^{N-1}, \ \sigma > \Delta_1; \ \text{and} \ |\eta_{KH}(\rho_1, \sigma)| < .5 - .25^{N-1}, \sigma < -\Delta_1. \end{array}$$

Since $|\Psi_k(\rho_1)| > w_1$, if the constant $w_2 < 1$ is positive and less than all $w_1 \exp d_1$'s

$$(7) \qquad |\Phi_{N-1,k}(\rho_1)| > w_2; \qquad (k = 1, \cdots, K); \ \text{see} \ (5).$$

To reduce the behavior of each F^*_N to that of a first order function, replace ρ^N by t to get the corresponding auxiliary function

$$(8) \qquad f(t, \rho, \sigma) = \sum_{k=1}^{K} \Phi_{N-1,k}(\rho) \exp \{2\pi b_{kN}(-N\sigma + it)\}.$$

Then in each auxiliary function set $\rho = \rho_1$ to get a finite set of analytic functions $f(t, \rho_1, \sigma)$ of the complex variable $w = -N\sigma + it$, regular throughout the entire w-plane. It is evident from (7) and (13) that when σ is large and positive the first term in (8) is dominant (i. e. largest in absolute value) and when σ is large and negative the last term is dominant. Therefore if

$$f(t, \rho_1, \sigma) = [1 + \eta_{kN}(t, \rho_1, \sigma)]\Phi_{N-1,k}(\rho_1) \exp \{2\pi b_{kN}(-N\sigma + it)\},$$

for all f there exists a positive constant Δ_2 such that

(9) $|\eta_{1N}(t, \rho_1, \sigma)| < 1/4$ when $\sigma > \Delta_2$ and $|\eta_{KN}(t, \rho_1, \sigma)| < 1/4$ when $\sigma < -\Delta_2$.

No $f(t, \rho_1, \sigma)$ vanishes when $|\sigma| > \Delta_2$ and the zeros of such a finite set of analytic functions are isolated. Hence there exist two positive constants t_0 and $w_3 < 1$ such that $|f(t_0, \rho_1, \sigma)| > 2w_3$ when $|\sigma| \leqq \Delta_2$.

Van der Corput's Theorem IV, p. 218 [4], guarantees the existence of a relatively dense set of values $t_1, t_2, \cdots$ which cause the quantities $b_{kN}t_j$, $(j = 1, 2, \cdots)$ associated with the various auxiliary f's, to simultaneously approximate, modulo 1, the respective values $b_{kN}t_0$ to within any preassigned accuracy ϵ. Since $|\sigma| \leqq \Delta_2$, the ϵ can and will be chosen small enough so that the $f(t_j, \rho_1, \sigma)$'s approximate, respectively, the $f(t_0, \rho_1, \sigma)$'s closely enough so that

(10) $\qquad |f(t_j, \rho_1, \sigma)| > w_3, \qquad |\sigma| \leqq \Delta_2, \qquad (j = 0, 1, 2, \cdots).$

Then considering simultaneously all terms of the form $b_{jm}\rho_1{}^m$ in the $\mathcal{P}_j$'s pertaining to all the $\Phi_{H-1,k}(\rho_1)$'s, and $\Phi_{N-1,k}(\rho_1)$'s, Van der Corput's Theorem IV [4] guarantees the existence of a relatively dense set of values ρ_i, $(i = 1, 2, 3, \cdots)$ which cause the respective $b_{jm}\rho_i{}^m$'s to simultaneously approximate, mod 1, the respective values $b_{jm}\rho_1{}^m$ to within any preassigned accuracy ϵ. Furthermore the ϵ can and will be chosen small enough so that the ρ_1 can be replaced by ρ_i in (6), (7), (9), and (10) without destroying the validity of these inequalities provided, at the same time, w_1 is replaced by $w_1/2$; w_2 by $w_2/2$; w_3 by $w_3/2$; $.5 - .25^{N-1}$ by $.5 - .5(.25)^{N-1}$; and $1/4$ by $3/8$.

If, however, smaller lower bounds and greater upper bounds were used in (6) through (10), these inequalities would remain valid for an even larger range of ρ values. More precisely, if w_1 is replaced by $w_1/4$, w_2 by $w_2/4$, w_3 by $w_3/4$, $.5 - .25^{N-1}$ by $.5 - .25^N$, and $1/4$ by $7/16$, all values of ρ in the intervals

$$(11) \qquad (\rho_i - \delta/\rho_i{}^{N-2}, \rho_i + \delta/\rho_i{}^{N-2}), \qquad (i = 0, 1, 2, \cdots)$$

are admissible in (6) through (10) if the positive constant δ is sufficiently small. These intervals decrease in length at a rate proportional to $1/\rho_i{}^{N-2}$ as $\rho_i \to \infty$.

When the relatively dense set of values $t_1, t_2, t_3, \cdots$ is mapped on the ρ-axis by the transformation $(t_i)^{1/N} = \rho'_i$, the distances between two successive points ρ'_i and ρ'_{i+1} decrease at a rate essentially proportional to $1/(\rho'_i)^{N-1}$. Hence any interval (11) sufficiently far out on the ρ-axis must contain at least one of the ρ'_i-points.

In other words there exists a relatively dense set of values such that

$$|\Phi_{H-1,k}(\rho'_i)| > w_1/4; \qquad\qquad |\Phi_{N-1,k}(\rho'_i)| > w_2/4;$$
$$|F^*_H(\rho'_i, \sigma)| > w_1/4, \ |\sigma| \leqq \Delta_1; \qquad |F^*_N(\rho'_i, \sigma)| > w_3/4, \ |\sigma| \leqq \Delta_2;$$
$$|\eta_{1H}(\rho'_i, \sigma)| < .5 - .25^N, \ |\sigma| > \Delta_1; \quad |\eta_{KH}(\rho'_i, \sigma)| < .5 - .25^N, \ \sigma < -\Delta_1;$$
$$|\eta_{1N}(\rho'_i, \sigma)| < 7/16, \ \sigma > \Delta_2; \quad \text{and} \quad |\eta_{KN}(\rho'_i, \sigma)| < 7/16, \ \sigma < -\Delta_2.$$

Therefore Theorem I is correct, provided it is true when $N = 1$, if Δ is the largest of the two quantities Δ_1 and Δ_2; and w is the smallest of the quantities

$$.25w_0[.5 + .25^N] \exp \{-2\pi N\Delta \, |\, b_{KM} \,|\}$$

where w_0 is the smallest of the three quantities $w_1/4$, $w_2/4$, $w_3/4$.

Theorem I can be demonstrated when $N = 1$ by repeating the reasoning given above; note particularly that all Φ_{ok} are constants if $N = 1$.

COROLLARY I. *Given a finite set of E^*-functions, there exist two positive constants w and B and a relatively dense set of values $\rho_0, \rho_1, \rho_2, \cdots$ such that for all functions F^* of the set*

$$|F^*(\rho_j, \sigma)| > w \exp \{-B \, |\, \sigma \,|\}, \qquad (j = 0, 1, 2, \cdots).$$

3. The variation in amplitude of $F^*_N(\rho, 0)$.

LEMMA L. 1 (taken from Bohr [3]). *Given two continuous complex valued functions $f_1(t)$ and $f_2(t)$ of the real variable t and four constants Ω, $w > 0$, $\epsilon < w$, and t_0 such that $\Omega > |f_1(t)| > w$, $|f_2(t)| > w$, and $|f_1(t) - f_2(t)| < \epsilon$ for all $t \geqq t_0$. Then, if the amplitudes $\phi_1(t)$ and $\phi_2(t)$ of f_1 and f_2, respectively, are defined as continuous functions of t, not only is*

$$(12) \qquad\qquad | f_1/|f_1| - f_2/|f_2| | < 2\Omega\epsilon/w^2;$$

but also, if $\Omega\epsilon/w^2 < 1$, there exists a definite integer g such that $|\phi_1(t) - \phi_2(t) - 2\pi g| < \pi\Omega\epsilon/w^2$ for all $t \geqq t_0$.

THEOREM II. *Given a complex valued function F^* of order N of the form*

$$(13) \qquad F^*(\rho) = \sum_{j=1}^{J} \exp\{c_j + 2\pi i(b_{jN}\rho^N + b_{j,N-1}\rho^{N-1} + \cdots + b_{j1}\rho)\}$$

where the c's are complex constants, the b's real constants, and the b_{jN}'s not all zero; if $F^*(\rho)$ is uniformly bounded away from zero for all $\rho \geqq 0$, then it follows that its amplitude $\Phi^*(\rho)$ defined as a continuous function of the real variable ρ has the form

$$(14) \qquad\qquad \Phi^*(\rho) = d_N\rho^N + d_{N-1}\rho^{N-1} + \cdots + d_1\rho + O(1)$$

where the d's are real constants.

Proof. If $N = 1$, refer to H. Bohr [3] for a demonstration. If $N > 1$, let $\Phi(t) = \Phi^*(t^{1/N})$. The same change of variable $\rho^N = t$ converts $F^*(\rho)$ into a new function $F(t) = F^*(t^{1/N})$, which for large values of t behaves much like an almost periodic function. More precisely given an $\epsilon > 0$, there exists a translation number $\tau = \tau(\epsilon) > 0$ and a constant $u = u(\epsilon) > 1$ such that $|F(t + \tau) - F(t)| < \epsilon$ if $t > u$.

Since the $|F|$ has both an upper bound Ω and a lower bound $w > 0$, there exists an integer g, see L. 1, such that

$$|\Phi(t + \tau) - \Phi(t) - 2\pi g| \leqq \pi\Omega\epsilon/w^2, \qquad t \geqq u.$$

The uniform convergence as $s \to \infty$ of the quotient $[\Phi(t + s) - \Phi(t)]/s$ to a limit d_N follows just as in [3]. Details need not be given. There is but one adjustment needed. If $t \leqq u$, set $s = n\tau + k$, n a large positive integer, $0 \leqq k \leqq \tau$, and use the identity

$$\Phi(t + s) - \Phi(t) = \begin{cases} [\Phi(t + s) - \Phi(u + s)] + [\Phi(u) - \Phi(t)] \\ \quad + [\Phi(u + s) - \Phi(u + n\tau)] \\ \quad + \sum_{v=1}^{n} [\Phi(u + v\tau) - \Phi(u + v\tau - \tau)]. \end{cases}$$

When $t > u$, use Bohr's identity, [3], p. 58.

The amplitude $\phi(t)$ of the function $f(t) = F(t) \exp\{-id_N t\}$ can now be evaluated and (14) established by mathematical induction on N. Note first that $\phi(t) = \Phi(t) - d_N t$ and that

$$(15) \qquad \lim_{s \to \infty} [\phi(t + s) - \phi(t)]/s = 0 \text{ uniformly for all } t \geqq 0.$$

An auxiliary function

$$f_1(t, s) = \exp\{-id_N t\} \cdot \sum_{j=1}^{J} \Psi_j(s) \exp(2\pi itb_{jN})$$

of the two independent variables t and s is introduced, where

$$\Psi_j(s) = \exp\{c_j + 2\pi i(b_{j,N-1}s^{(N-1)/N} + \cdots + b_{j1}s^{1/N})\}.$$

It is almost periodic in t and to a positive $\epsilon < w^2/48\Omega(J+2)$ there corresponds an infinite set of translation numbers τ_n and an inclusion interval $L > 1$ with $nL \leqq \tau_n < L(n+1)$ for $(n = 1, 2, 3, \cdots)$. The L and the τ_n's are independent of s.

Utilizing the inequalities $|1 - \exp\{i\alpha\}| \leqq |\alpha|$, α real, and $(s + \sigma)^{q/N} - s^{q/N} \leqq \sigma/s^{1/N}$, for $(q = 1, 2, \cdots, N-1)$; $s \geqq 1$; $s \geqq \sigma \geqq 0$, it is clear there exists a constant G independent of s, j, and σ such that

$$(16) \qquad\qquad |\Psi_j(s + \sigma) - \Psi_j(s)| < \sigma G/s^{1/N}.$$

Hence, if G is so chosen that $\eta = (2GL/\epsilon)^N/L$ is an integer, then when $n \geqq \eta$

$$(17) \qquad |\Psi_j(\tau_{n+1} + t) - \Psi_j(\tau_n + t)| < \epsilon; \qquad\qquad (j = 1, \cdots, J; \ t \geqq 0).$$

Since f_1 is almost periodic

$$(18) \qquad |f_1(t + \tau_n, s) - f_1(t, s)| < \epsilon; \qquad (n = 1, 2, \cdots; \ t \text{ and } s \geqq 0).$$

Because of (17) the $|f(\tau_{n+1} + t) - f_1(\tau_{n+1} + t, \ \tau_n + t)| < \epsilon J$. Using this result and (18) twice, it is found that

$$(19) \qquad |f(\tau_{n+1} + t) - f(\tau_n + t)| < \epsilon(J + 2), \quad n \geqq \eta, \quad t \geqq 0.$$

Since $|f(t)|$ has the same upper and lower bounds as has $|F(t)|$, to each $n \geqq \eta$ there corresponds an integer g_n such that

$$(20) \qquad |\phi(\tau_{n+1} + t) - \phi(\tau_n + t) - 2\pi g_n| < \pi/3, \qquad \text{see L. 1.}$$

But all these g_n's are zero: for suppose $g_i \neq 0$, $i \geqq \eta$; then either $\{\phi(\tau_{i+1} + t) - \phi(\tau_i + t)\} > \pi$ or $< -\pi$ for all $t \geqq 0$. In both cases

$$|\phi(\tau_i + m[\tau_{i+1} - \tau_i]) - \phi(\tau_i)| > m\pi,$$

m an integer, and

$$\lim_{m \to \infty} |\{\phi(\tau_i + m[\tau_{i+1} - \tau_i]) - \phi(\tau_i)\}/m[\tau_{i+1} - \tau_i]| > \pi/(\tau_{i+1} - \tau_i) > 0$$

contradicting (15). Consequently

$$(21) \qquad\qquad |\phi(\tau_{n+1}) - \phi(\tau_n)| < \pi/3 \qquad\qquad \text{for all } n > \eta.$$

Furthermore $f(t)$ and $\phi(t)$ are uniformly continuous on the interval $0 \leq t < \infty$, just as were F and Φ. Hence there exists a constant B independent of n such that

$$(22) \qquad |\phi(t) - \phi(\tau_n)| < B \quad \text{when } \tau_n \leq t \leq \tau_{n+1}, \qquad (n = 1, 2, 3, \cdots).$$

Plot the values of $f(\tau_n)$ for $n = \eta,\ \eta + 1,\ \eta + 2, \cdots$ in a complex f-plane, connecting each point $f = f(\tau_n)$ to the next $f = f(\tau_{n+1})$ by a straight line segment. This infinite chain $\mathcal{D}_2$ of connected line segments, while only roughly portraying the variation of $f(t)$ as $t \to \infty$, exactly portrays the fluctuation of the function

$$f_2(t) = f(\tau_n) + [t - \tau_n][f(\tau_{n+1}) - f(\tau_n)]/(\tau_{n+1} - \tau_n), \quad \tau_n \leq t \leq \tau_{n+1},\ n \geq \eta.$$

Since $|f(t)| > w$, $|f_2(t)| > w/2$, see (21).

Let the amplitude of $f_2(t)$ be $\phi_2(t)$, defining it as a continuous function of t with $\phi_2(\tau_\eta) = \phi(\tau_\eta)$. Then $\phi_2(\tau_n) = \phi(\tau_n)$ for all $n \geq \eta$, for as t runs from τ_n to τ_{n+1}, $f_2(t)$ traces out one segment of $\mathcal{D}_2$ and neither $\phi_2(t)$ or $\phi(t)$ can vary by as much as π, see (21). Therefore (22) can be replaced by

$$(23) \qquad\qquad |\phi(t) - \phi_2(\tau_n)| < B,\ \tau_n \leq t \leq \tau_{n+1},\ \text{and } n \geq \eta.$$

Also

$$(24) \qquad\qquad |\phi_2(t) - \phi_2(\tau_n)| < \pi \quad \text{for } \tau_n \leq t \leq \tau_{n+1}\ \text{and } n \geq \eta.$$

To use induction, convert $f_1(\tau_\eta, t)$ into a function of ρ by setting $t = \rho^N$. As a function of ρ, $f_1(\tau_\eta, \rho^N)$ is of type (13) and is of order less than N. Note first that $F(t)$ and $f(t)$ both have the same upper and lower bounds and hence $\Omega > |f(\tau_n)| > w > 0$ for all n. Secondly a double application of (18) shows that

$$(25) \qquad\qquad\qquad |f_1(\tau_\eta, \tau_n) - f(\tau_n)| < 2\epsilon$$

for all n; and thirdly as a consequence of (16),

$$(26) \qquad |f_1(\tau_\eta, \tau_n) - f_1(\tau_\eta, t)| < \epsilon J \quad \text{if } \tau_n \leq t \leq \tau_{n+1}\ \text{and } n \geq \eta.$$

These three facts combined make it clear that $w/2 < |f_1(\tau_\eta, t)| < 2\Omega$ if $t \geq \tau_\eta$. Thus Theorem II becomes applicable by hypothesis and the amplitude $\phi_1(t)$ of $f_1(\tau_\eta, t)$, defined as a continuous function of t, is given by the equation

$$\phi_1(t) = d_{N-1} t^{(N-1)/N} + \cdots + d_1 t^{1/N} + O(1).$$

When a point representing the complex number $f_1(\tau_\eta, \tau_n)$, $n = \eta,\ \eta + 1$,

$\eta + 2, \cdots$, is marked in the complex f-plane, its distance from the point $f(\tau_n)$ is not as great as 2ϵ, see (25). As the continuous set of values taken on by $f_1(\tau\eta, t)$ is traced out in the f-plane as t varies from $\tau\eta$ to ∞, a continuous curve $\mathcal{D}_1$ is generated which differs very little from $\mathcal{D}_2$. More explicitly

$$(27) \qquad |f_2(t) - f_1(\tau\eta, t)| < \epsilon(4 + 2J), \qquad t \geqq \tau\eta.$$

This inequality is a consequence of (25), (26), and the fact that, by (19), the segments of $\mathcal{D}_2$ do not exceed $\epsilon(2 + J)$ in length.

From (27) and L. 1 it is obvious that there exists an integer g such that $|\phi_2(t) - \phi_1(t) - 2\pi g| < \pi/3$ for $t \geqq \tau\eta$. This result combined with (24) and (23) yields $|\phi(t) - \phi_1(t)| < B + \pi(2g + 4/3)$, $t \geqq \tau\eta$ and completes the demonstration of Theorem II.

4. Preliminary notation. Returning to functions of type (1), let

$$(28) \qquad q_j = \sum_{k=1}^{K} a_{jk} z^k$$

and $P_j = c_j z^{v_j} +$ terms of lower degree, with the a's and c's complex constants; $c_j \neq 0$; and at least one of the a_{jK}'s $\neq 0$.

Then call any function of structure (1), (28) an *E-function*. If $J = 1$, the E-function is, by definition, of *order* zero and of *degree* K. If $J > 1$, the *order* is N and the *degree* K provided:

I) There are at least two a_{jN}'s having different values.

II) No two of the polynomials q_j are identical.

III) The $a_{1k} = a_{2k} = \cdots = a_{Jk}$ for $k = N + 1, N + 2, \cdots, K$ if $N < K$.

When the zeros of a particular E-function are to be located, remove an exponential factor which will reduce the degree of the function to that of the order. The number and location of the zeros remains unchanged and $E(z)$ is replaced by

$$(29) \qquad E_{-1}(z) = \sum_{j=1}^{J} P_j \exp Q_j \quad \text{with} \quad Q_j = \sum_{n=1}^{N} a_{jn} z^n.$$

The ξ-curves which are used are given in polar coördinates by equations of the form

$$(30) \qquad \theta = M_0 + M_1/\rho + \cdots + M_s/\rho^s + M_{s+1} \log \rho/\rho^{s+1} + M_{s+2}/\rho^{s+1}$$

where $z = \rho \exp\{i\theta\}$. The M's are real constants adjusted at pleasure as portions of the z-plane are explored.

To compute the magnitude of a typical term $P \exp Q$ on ξ, (the leading subscript j is temporarily omitted on all symbols), begin by separating $a_n z^n$ into its real and imaginary parts:

(31) $\mathcal{R}(a_n z^n) = \rho^n d_n \cos(n\theta - \alpha_n)$; $\mathcal{I}(a_n z^n) = \rho^n d_n \sin(n\theta - \alpha_n)$

where $d_n = |a_n|$ and α_n is the argument of $\bar{a}_n$, the conjugate of a_n. Let

$$\mathcal{R}(\log c) = h \quad \text{and} \quad \mathcal{I}(\log c) = \mu, \quad 0 \leqq \mu < 2\pi.$$

Then the modulus $\mathcal{M}$ and argument $\mathcal{A}$ of $P \exp Q$ take the form

(32) $\mathcal{M} = \exp\{\epsilon(\rho, \theta) + h + v \log \rho + \sum_{n=1}^{N} \rho^n d_n \cos(n\theta - \alpha_n)\}$

and

(33) $\mathcal{A} = \epsilon(\rho, \theta) + \mu + v\theta + \sum_{n=1}^{N} \rho^n d_n \sin(n\theta - \alpha_n).$

The ϵ-functions here and in subsequent formulas uniformly approach zero as $z \to \infty$.

If z is on a ξ-curve with $s = N - 1$,

(34) $\log \mathcal{M} = A_0 \rho^N + \cdots + A_{N-1}\rho + A_N \log \rho + A_{N+1} + h + \epsilon(\rho)$

and

(35) $\mathcal{A} = B_0 \rho^N + \cdots + B_{N-1}\rho + B_N \log \rho + B_{N+1} + v M_0 + \mu + \epsilon(\rho)$

where the A's and B's are functions of the M's which are independent of ρ.

To exhibit this functional relationship write

(36) $X_{N-n} = N d_n \sin(n M_0 - \alpha_n)$; $Z_{N-n} = N d_n \cos(n M_0 - \alpha_n)$

for $n = 1, \cdots, N$; and let the X's and Z's be zero for $n = -1$ and 0. Then the A's of (34) become functions of the X's, Z's, and M_i's; $i > 0$. In fact

(37) $A_0 = Z_0/N$ and $A_k = Y_k - M_k X_0$; $(k = 1, \cdots, N+1)$;

where $Y_N = v$ and

(38) $Y_k = V_k(M_1, \cdots, M_{k-1}; X_1, \cdots, X_{k-1}; Z_0, \cdots, Z_{k-1}) + Z_k/N,$
$$(k = 1, \cdots, N-1, N+1).$$

Each V is a polynomial in the indicated variables. The values of a particular

Y need not depend upon all the variables listed; for instance Y_{N+1} is independent of M_N. Also if $N > 1$, $V_1 \equiv 0$.

To procure formulas for the B's of (35) analogous to (37) and (38) replace in (37) and (38) each A by B, each Z by X, and each X by $(-Z)$ when $k = 0, 1, \cdots, N - 1$ and $N + 1$. The $B_N = M_N Z_0$.

When $|z|$ is large, it is evident from (34) that the relative sizes of the terms of E_{-1} depend primarily upon the quantities

$$A_{j0} = d_{jN} \cos(N M_0 - \alpha_{jN}).$$

The leading subscript j reappears in order to distinguish the quantities pertaining to one term from those pertaining to another.

5. Dominant terms in g_i-strips. The zero distribution for E_{-1} is closely related to certain critical ξ_i-curves in the z-plane. Once these ξ_i's are located each in turn is covered by a g_i-strip bounded on the left and on the right by two curves

$$(39) \qquad \theta = \Theta_i \pm \delta_i/\rho^i \text{ with } \Theta_i = m_0 + m_1/\rho + \cdots + m_i/\rho^i$$

where the m's are real constants, presently to be specified, and δ_i is a positive constant controlling the width of the strip, arbitrarily selected, but small enough to satisfy certain requirements $\Re_{ik}$, $k = 1, 2, 3$. In particular the entire z-plane exterior to the C_0 circle is called a g_{-1}-strip.

To each g_i there corresponds an E-function E_i of order N. Each E_i will be formed from E_{-1} by deleting appropriate terms. The m's of (39) will be so chosen that the A_{jk}'s pertaining to the terms of E_i become equal to the respective constants A_k for $k = 0, 1, \cdots, i$ when the M_k's of (30) take on the values of the respective m_k's of (39). This means that the relative sizes of the terms of E_i are primarily controlled by the quantities

$$(40) \qquad A_{j, i+1} = Y_{j, i+1} - M_{i+1} X_{j0}$$

when z is on the curve

$$\Xi: \quad \theta = \Theta_i + M_{i+1} \boldsymbol{\mathcal{L}}_{i+1}/\rho^{i+1}$$

located in g_i. The symbol $\boldsymbol{\mathcal{L}}_i = 1$ if $i < N$ and $= \log \rho$ if $i = N$. Note also that as z traces out the curve Ξ the X's and Y's of (40) do not vary with either ρ or M_{k+1}, since $M_k = m_k$, $k = 0, 1, \cdots, i$; see (36) and (38).

To sort the A_{j0}'s of E_{-1} according to size, begin by plotting the points a_{jN} in the complex z-plane. Then, using these points, draw the primary critical

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polygon D_0, as well as each primary critical ray ξ_0. These rays radiate from the origin and make with the positive real axis the respective angles

(41) $- (\phi_a + 2\beta\pi)/N$, $(\alpha = 0, \cdots, M'; \beta = 0, \cdots, N - 1)$;

see Mac Coll [1], p. 343, for the meaning of the symbols in (41) and details on D_0 and ξ_0.

Each ξ_0 is then to be covered by a g_0. This means that in (39) the appropriate values for m_0 are the angular values of (41). δ_0 is chosen sufficiently small, in accordance with $\Re_{01}$, so that no two g_0's overlap or have a common boundary. The sectors of the z-plane which lie between the consecutive g_0's are labelled p_0-regions.

To sort the $A_{j,i+1}$'s of (40) pertaining to the E_i of a particular g_i, $i \geqq 0$, plot a point with coördinates $(X_{j0}, Y_{j,i+1})$ corresponding to each term of E_i in an XY-rectangular Cartesian system. Then, using these points, draw the associated Newton or Puiseux diagram [3] D_{i+1} and note the respective slopes $m_{i+1,1}, \cdots, m_{i+1,h}$ of the successive sides of D_{i+1}, numbering from left to right. The critical curves ξ_{i+1} of E_{-1} in g_i are defined by the equations

$$\theta = \Theta_i + m_{i+1,k}\mathcal{L}_{i+1}/\rho^{i+1}; \qquad\qquad (k = 1, \cdots, h).$$

Each ξ_{i+1} is covered by a g_{i+1} bounded on the left and on the right by two curves

$$\theta = \Theta_i + (m_{i+1,k} \pm \delta_{i+1})\mathcal{L}_{i+1}/\rho^{i+1}.$$

δ_{i+1} is chosen small enough so that in accordance with $\Re_{i+1,1}$ no two g_{i+1}-strips overlap or have a common boundary. Those portions of the g_i which remain after the g_{i+1} strips are marked off are labelled p_{i+1} regions.

To each term of E_i, $(i = -1, 0, 1, \cdots, N - 1)$, corresponds a plotted point, to each g_{i+1} a corresponding side, and to each p_{i+1} a corresponding vertex of D_{i+1}. Single out a particular g_{i+1} and the corresponding side L_{i+1} in D_{i+1}. Then it is evident from D_{i+1} that, if δ_{i+1} is chosen sufficiently small, the terms of E_i which become largest (or dominant) in g_{i+1} are those corresponding to plotted points on L_{i+1}. It will therefore be assumed, as $\Re_{i+1,2}$ demands, that δ_{i+1} is chosen small enough at the outset to bring this dominance into effect. Moreover the dominance is so strong that, if T'_{i+1} represents the sum of the absolute values of the terms of E_i which do not correspond to plotted points on L_{i+1}, and T_{i+1} represents any one of the dominant terms

(42) $| T'_{i+1}/T_{i+1} | < \exp\{- B\rho^{N-i-1}\mathcal{L}_{i+1}\}$ in g_{i+1}.

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[3] For details see Langer [5], bottom p. 222.

Here, and in succeeding inequalities, B is an appropriate positive constant. It is also tacitly assumed that ρ_0 has been chosen sufficiently large at the outset so that all inequalities containing B's are valid on and beyond the C_0-circle.

The function E_{i+1} correspond to $\dot{g}_{i+1}$ is formed from E_i by deleting from E_i all terms which are not dominant in g_{i+1}. The terms of E_{i+1} have been so chosen that if a particular T_{i+1} is selected; $F_{i+1} = E_{i+1}/T_{i+1}$; and z is confined to g_{i+1}

$$(43) \qquad E_i = T_{i+1}(F_{i+1} + \eta_{i+1}) \text{ with } |\eta_{i+1}| < \exp\{-B\rho^{N-i-1}\boldsymbol{\mathcal{L}}_{i+1}\},$$
$$(i = -1, 0, 1, \cdots, N-1).$$

Consequently, if z is in g_{i+1}, $k = -1, 0, \cdots$, or i, and $i = 0, 1, \cdots$, or $N-1$

$$(44) \quad E_k = T_{i+1}(F_{i+1} + \eta_{i+1,k}) \text{ with } |\eta_{i+1,k}| < \exp\{-B\rho^{N-i-1}\boldsymbol{\mathcal{L}}_{i+1}\}.$$

6. Dominant terms in p_i-regions. With the g_i's located, the E_i's defined, and the associated dominances recorded, turn to the p_i's. Consider a particular *interior* p_i which is sandwiched in between two g_i's and note that it is bounded on the left and right by the two curves

$$(45) \quad \theta = \Theta_{i-1} + (m_{ik} - \delta_i)\boldsymbol{\mathcal{L}}_i/\rho^i \text{ and } \theta = \Theta_{i-1} + (m_{i,k+1} + \delta_i)\boldsymbol{\mathcal{L}}_i/\rho^i;$$

$i = 1, \cdots, N$. Then glance at the appropriate D_i and pick out the vertex V corresponding to p_i. It is obvious from D_i that the largest (or dominant) terms t_i of E_{i-1} in p_i are those which correspond to plotted points falling on V. The dominance is so strong that the

$$(46) \qquad\qquad |t'_i/t_i| < \exp\{-B\rho^{N-i}\boldsymbol{\mathcal{L}}_i\}$$

where t'_i represents the sum of the absolute values of the terms of E_{i-1} not corresponding to V. Let $\boldsymbol{\Psi}_i$ be the sum of the t_i terms; choose a particular t_i and set $f_i = \boldsymbol{\Psi}_i/t_i$; then as a consequence of (44) and (46) in p_i

$$(47) \qquad E_{k-1} = t_i(f_i + \eta_{ik}) \text{ with } |\eta_{ik}| < \exp\{-B\rho^{N-i}\boldsymbol{\mathcal{L}}_i\}$$

for $k = 0, 1, \cdots$ or i and $i = 1, 2, \cdots$ or N. The newly defined function $\boldsymbol{\Psi}_i$ is an E-function of order less than N since all its terms correspond to a single point in each of the D_k-diagrams, $k = 0, 1, \cdots, i$.

On the other hand a p_i region on the extreme left (or right) in a g_{i-1} requires special attention for it is bounded by the two curves

$$\theta = \Theta_{i-1} + \delta_{i-1}/\rho^{i-1} \text{ and } \theta = \Theta_{i-1} + (m_{i1} + \delta_i)\boldsymbol{\mathcal{L}}_i/\rho^i$$

and *the first of these curves is not of type* (45). It might be expected that all dominant terms t_i of E_{i-1} in this p_i would correspond to the point V_L at the extreme left on D_i and that the individual terms t'_i not corresponding to V_L would be dominated to such an extent in p_i that (46) and (47) would be valid. But the following computations show that (46) and (47) may not be valid unless δ_{i-1} satisfies a third restriction $\Re_{i-1,3}$ in addition to $\Re_{i-1,1}$ and $\Re_{i-1,2}$.

The necessity and nature of this new restriction becomes evident as an appropriate upper bound on the $\log | t'_i/t_i |$ is computed. For this purpose confine z to the curve

(48) $$\theta = \Theta_{i-1} + m/\rho^{i-1}$$

and let m vary over the finite range

(49) $$(m_{i1} + \delta_i)\,\mathcal{L}_i/\rho \leqq m \leqq \delta_{i-1}.$$

This variation in m makes the curve (48) sweep out the entire p_i-region. On (48)

(50) $$\log | t'_i/t_i | = (A_{10} - A_{00})\rho^N + (A_{11} - A_{01})\rho^{N-1} + \cdots,$$

where the leading subscript on each A is set equal to zero when the A pertains to t_i and set equal to one when it pertains to t'_i.

As long as z is in p_i the M's of (30) are related to the m's of (48) as follows:

$$M_k = m_k \text{ for } (k = 0, 1, \cdots, i-2)\,; M_{i-1} = m_{i-1} + m \text{ and } M_k = 0 \text{ for } k \geqq i.$$

Consequently $A_{1k} - A_{0k} = 0$ for $k = 0, 1, \cdots, i-2$ and the first $(i-1)$ terms of (50) vanish. If the A's in the remaining non-vanishing terms are replaced by equivalent expressions in the Y's and X's, see (37) and (38), the leading non-vanishing term of (50) becomes $m(X_{00} - X_{10})\rho^{N-i+1}$. The leading term takes this abbreviated form because all terms of E_{i-1} correspond to collinear points in D_{i-1} and therefore

$$Y_{1,i-1} - Y_{0,i-1} - m_{i-1}(X_{10} - X_{00}) = 0.$$

The expansion (50) is further condensed by writing

$$y_j(m) = V_i(m_1, \cdots, m_{i-2}, m_{i-1} + m;\ X_{j1}, \cdots, X_{j,i-1};\ \cdots, Z_{j,i-1}) + Z_{ji}/N.$$

This abbreviation emphasizes that m is the only variable when (48) is sweeping out p_i.

Once all these simplifications are made and it is noted that $X_{10} \geqq X_{00}$, it is apparent that in p_i

$$(51) \qquad \log | t'_i/t_i | < \boldsymbol{\mathcal{L}}_i \rho^{N-i} \{ W(m) - \delta_i(X_{10} - X_{00}) + \epsilon(\rho, m) \}$$

where

$$W(m) = y_1(m) - y_0(m) - m_{i1}(X_{10} - X_{00}).$$

To cast this upper bound on the log into the desired form, note that there are three possible locations in the D_i-diagrams for the point corresponding to t'_i: [I] it may be to the right of vertex V_L and on the segment of slope m_i; [II] it may be to the right of V_L, but not on the segment of slope m_{i1}; or [III] it may be directly below V_L. In case [I] $W(0) = 0$, $X_{10} > X_{00}$ and consequently $\delta_i(X_{00} - X_{10})$ is definitely negative and less than a constant $(-B)$; in case [II] $W(0)$ is negative and $\delta_i(X_{00} - X_{10})$ is less than some constant $(-B)$; and in [III] $X_{00} = X_{10}$ and $W(0)$ is negative and less than some constant $(-B)$. Hence in all three cases

$$W(m) - \delta_i(X_{10} - X_{00}) + \epsilon(\rho, m)$$

will remain less than a $(-B)$ provided the range (49) is sufficiently restricted to keep the values of the continuous function $W(m)$ close enough to $W(0)$. Therefore δ_{i-1} can and, in accordance with requirement $\mathfrak{R}_{i-1,3}$, will be chosen small enough and ρ_0 large enough so that throughout p_i

$$\log | t'_i/t_i | < - B\boldsymbol{\mathcal{L}}_i \rho^{N-i}.$$

It is clear from this analysis that δ_{i-1} depends upon δ_i. Therefore in selecting a set of δ's satisfying all the $\mathfrak{R}$-requirements, δ_N is chosen first, subject to two restrictions, $\mathfrak{R}_{N1}$ and $\mathfrak{R}_{N2}$, next δ_{N-1} is chosen subject to three restrictions, then δ_{N-2}, and so on.

7. Subdivisions of a g_N-strip. Select any particular g_N-strip and the associated side S of slope m_N in the appropriate D_N-diagram and split the strip into three parts: two side regions labelled p_{N+1} and a central g_{N+1}-strip bounded on the left and right by the two curves

$$\theta = \Theta_{N-1} + (m_N \log \rho \pm \Delta_N)/\rho^N$$

where Δ_N is any sufficiently large positive constant satisfying a requirement $\mathfrak{R}_N$. The nature and necessity of this requirement will appear in the proof of Theorem III.

Note that in the p_{N+1} on the left the dominant or largest terms t_i of E_N

are those which correspond to the vertex V_L at the left end of S. The dominance is such that, if t'_{N+1} is the sum of the absolute values of the terms of E_N not corresponding to V_L,

$$(52) \qquad\qquad | t'_{N+1}/t_{N+1} | < b$$

where b is an arbitrary constant less than unity. Further b can be made as small as desired by selecting Δ_N and ρ_0 sufficiently large. The computations substantiating (52) are analogous to those already given for other p_i-regions. In this instance p_{N+1} is swept out by the curve $\theta = \Theta_{N-1} + (m_N + m)\log \rho/\rho^N$ as m varies over the range $\Delta_N/\log \rho \leqq m \leqq \delta_N$; and the inequality corresponding to (51) is

$$(53) \quad \log | t'_{N+1}/t_{N+1} | \leqq h_1 - h_0 - \Delta_N(X_{10} - X_{00}) + Y_{1,N+1} - Y_{0,N+1} + \epsilon(\rho, m)$$

with the h's and Y's fixed. Since in this case $X_{10} > X_{00}$, it is evident that a sufficiently large Δ_N and ρ_0 can be selected so as to make the right member of (53) as large a negative number as desired. Consequently (52) holds throughout the left p_{N+1}-region. Analogous results hold for the p_{N+1}-region on the right.

The dominant t_{N+1} terms of E_N are added to form the function Ψ_{N+1}. A particular term t_{N+1} of Ψ_{N+1} is singled out. Then with

$$f_{N+1} = \Psi_{N+1}/t_{N+1}$$

(52) implies that in p_{N+1}

$$(54) \quad E_{k-1} = t_{N+1}(f_{N+1} + \eta_{N+1,k}) \text{ with } | \eta_{N+1,k} | < b < 1, \quad (k = 0, 1, \cdots, N+1).$$

Since all terms of Ψ_{N+1} correspond to the same point in each of the respective $D_0, D_1, \cdots, D_N$ diagrams, Ψ_{N+1} is an E-function of order less than N. Moreover (54) shows that in p_{N+1} the behavior of E_{-1} is essentially the same as that of the lower ordered E-function Ψ_{N+1}. Thus we find that *the given E-function of order N behaves like an E-function of order less than N in all portions of the z-plane, except in the g_{N+1}-strips.* To discover the behavior of E_{-1} in these g_{N+1}-strips, the amplitudes as well as the moduli of the individual terms must eventually be taken into account.

A particular g_{N+1} is swept out by a curve

$$(55) \qquad\qquad \theta = \Theta_{N-1} + (\sigma + m_N \log \rho)/\rho^N$$

as σ varies from $-\Delta_N$ to Δ_N. In this region

$$(56) \qquad E_{-1} = T_N(F_N + \eta_N) \text{ with } | \eta_N | < \rho^{-B}, \text{ see } (44).$$

Moreover all terms of $E_N = T_N F_N$ have been so chosen that in g_{N+1} the

A_{j0}'s are all alike. The same is true for the A_{j1}'s, the A_{j2}'s,$\cdots$, and the A_{jN}'s. Hence when E_N is divided by T_N all the A's disappear and in g_{N+1} F_N takes on the special structure

$$(57) \quad F_N(z) = F_N(\rho, \sigma) = \sum_{j=1}^{J} \exp\{[d_j - 2\pi N b_{jN}\sigma] + 2\pi i [\mathcal{P}_j(\rho) + b_{j0}] + \epsilon(\rho, \sigma)\}$$

where the d's and b's are certain, known, real constants and

$$(58) \qquad \mathcal{P}_j(\rho) = b_{jN}\rho^N + b_{j,N-1}\rho^{N-1} + \cdots + b_{j1}\rho.$$

LEMMA 2. *In* (57) *no two of the polynomials* $\mathcal{P}_j$ *are identical and at least two of the* b_{jN}'s *are distinct.*

Proof. Since all terms of $E_N = T_N F_N$ correspond to a specific side S in a D_N, there is at least one term T_1 corresponding to the left end of S and another T_2 to the right end. The b_{jN}'s for T_1/T_N and T_2/T_N are necessarily different for $X_{10} \neq X_{20}$.

If two polynomials $\mathcal{P}_1$ and $\mathcal{P}_2$ in any two terms T_1/T_N, T_2/T_N were identical the corresponding A's and B's of (34) and (35) for T_1 and T_2 would be equal. As a consequence the X's and Z's of (36) would be equal and therefore the corresponding a's of (29) would be equal. But the Q's can not be equal for this is in direct contradiction with stipulation II, 4 defining an E-function, therefore no two $\mathcal{P}$-polynomials are identical.

$\cdot$ It is conceivable that for a given σ, say σ_0, $F_N(\rho, \sigma_0)$ is *uniformly bounded away from zero* for all large ρ. If this be true, it immediately follows from continuity considerations that there exist two positive constants δ, w such that $|F_N(z)| > w > |\eta_{N+1}|$ if z lies in the strip bounded by the two curves

$$\theta = \Theta_{N-1} + (\sigma_0 \pm \delta + m_N \log \rho)/\rho^N.$$

Moreover it is evident from (56) that *in this same strip* E_{-1} *can not vanish and that a zero-free* W_N-*region will have been found.*

To investigate the situation when $\sigma = \sigma_0$ draw the lines

$$y = \exp\{d_j - 2\pi N b_{jN}x\}, \qquad\qquad (j = 1, \cdots, J),$$

on semi-log paper to see if the largest value of y at $x = \sigma_0$ exceeds the sum of all other y values. If this is true, $F_N(\rho, \sigma_0)$ is bounded away from zero for all large ρ and a W_N can be marked off as indicated.

To locate a W_N in this way, it is not necessary that there be a single term in dominance, but it is essential that when $\sigma = \sigma_0$ and $\rho \to \infty$ there be a lower bound w for the $|F_N(\rho, \sigma)|$ which is different from zero. The author has a

method for computing w for any given σ_0 and F_N, but as it is long, no details will be given in this paper.

Those portions of a g_{N+1} that are not definitely known to be W_N-regions are marked Z-strips. A g_{N+1} will therefore be subdivided into a finite number of regions, alternately marked Z_N and W_N. If a g_{N+1} contains no W_N's, or if no time has been taken to locate such regions, the entire g_{N+1} is marked a Z_N-strip.

8.   The Z and W-regions of an E-function. With the W_N and Z_N regions located and marked, the remaining unmarked portions of the z-plane can now also be subdivided into Z and W-regions. Note first that the W and Z-regions for E will be the same as those for E_{-1} for the zeros of the two functions are identical, both as to location and multiplicity. Observe also that since a zeroth-ordered E is the product of a polynomial and a non-vanishing exponential and since the C_0-circle can and will be chosen large enough at the outset to circumscribe all the zeros of the polynomial, the entire z-plane on and beyond C_0 is a zero-free W-region for a given zeroth-ordered E-function.

If an E of order $N > 0$ is given, form the corresponding E_{-1} and locate the zero-free W_N and Z_N-regions for the E_{-1} as explained in **7**. Then label each of the unmarked portions of the z-plane a P_s-sector of E_{-1}. These sectors will be sandwiched in between successive g_{N+1}'s and will be, in general, composite, for each P_s is made up of an odd number of adjacent p_i's, arranged in the following symmetric order:

$$(59) \qquad p_{N+1}, p_N, \cdots, p_{s+1}, p_s, p_{s+1}, \cdots, p_N, p_{N+1}; \qquad (s = 0, 1, \cdots, N).$$

With each of these p_i's there is associated a definite E-function $\Psi_i = t_i f_i$ of order $H < N$, see (47) and (54). The particular Ψ which corresponds to the central p_s of (59) is especially important for it will be found that E_{-1} *and Ψ_s behave in essentially the same way throughout P_s.* For this reason we introduce the following

DEFINITION. *The Z and W-regions, or portions there of, which pertain to Ψ_s and are located in P_s are the respective Z and W-regions for E_{-1}, as well as for E.*

In defining Z's and W's of an E of order N in this way, it is presupposed that Z and W-regions of certain E's of lower order have been located in advance. Such a definition is justified, for Theorem III makes it clear that these W's are actually zero-free. Before stating Theorem III, it is necessary to associate with the given E an appropriate relatively dense set of numbers

$\rho_0, \rho_1, \rho_2, \cdots$. These ρ_i's are radii of the C_i-circles, $|z| = \rho_i$, which will frequently be used as the analysis progresses.

To locate these circles write down the E_{-1} corresponding to E; draw the D-diagrams and cast out insignificant terms to get the F_N functions corresponding to the g_{N+1} regions. Then erase the $\epsilon(\rho, \sigma)$'s in (57) to form the corresponding F^*_N's. In this way there is associated with E a definite set of F^*_N's.

Put these F^*_N's momentarily aside and turn to the Ψ's, which correspond to the respective central p_s-regions of the P_s-sectors of E. These Ψ_s are E's of orders $H < N$. The process is repeated. As the F^*_N were associated with, and derived from, E so there is also associated with, and derived from, each of these Ψ_s's of order $H > 0$ a set of F^*_H functions. These F^*_H's are also temporarily put aside and adjoined to the F^*_N's.

As the Ψ_s were associated with E, so there is also associated with each Ψ_s of order $H > 0$ a set of Ψ_G's. These Ψ_G's correspond to the central p-regions of the P_s-sectors of Ψ_s and are E's of order $G < H$. The Ψ_G's of order $G > 0$ are treated just as were the E and Ψ_H's of order $H > 0$. F^*_G's are set aside, etc. Thus the process continues and will end when F^*_N's, F^*_H's, F^*_G's, $\cdots$, and F^*_1's have been set aside. According to Corollary I there is associated with this set of functions a relatively dense set of values $\rho_0, \rho_1, \rho_2, \cdots$ and two positive constants w and B such that for all F^*_j's of the set

$$(60) \quad |F^*_j(\rho_i, \sigma)| > w \exp\{-B|\sigma|\}; \quad (i = 0, 1, \cdots; j = 1, \cdots, N).$$

These ρ's are the desired radii for the C_i-circles.

Once the Δ_1's, $\cdots$, Δ_G's, Δ_H's, Δ_N's are chosen, the widths of the strips $g_2, \cdots, g_{G+1}, g_{H+1}, g_{N+1}$ corresponding to the respective functions $\Psi_1, \cdots, \Psi_G$, Ψ_s, E are fixed. This in turn regulates the allowable σ variation, see (55), and with $\rho \to \infty$ the respective F_N's, F_H's, F_G's, $\cdots$ approach uniformly the F^*_N's, F^*_H's, $\cdots$. In view of (60) this leads to

COROLLARY II. *The functions* F_N; F_H, F_G, $\cdots$, F_1 *associated with a particular E-function of order* N *are all uniformly bounded away from zero in their respective strips* $g_{N+1}, g_{H+1}, g_{G+1}, \cdots, g_2$ *if* z *is on any* C_i-circle.

Given an E, the entire z-plane exterior to the C_0-circle can be cut up, as previously indicated, into a finite number of Z and W-regions. Let $\mathcal{S}$ be any particular one of these regions, Ψ the sum of the terms of E dominant in $\mathcal{S}$, T any particular term of Ψ, $F = \Psi/T$, and t' the sum of the absolute values of the terms of E not in Ψ. Then by

THEOREM III. *There exist two positive constants* $w < 1$ *and* b *such that*

(61) $E = T(F + \eta)$ and $|\eta| \leqq |t'/T| < b < w < |F(z)|$

if z is in a W-region, or if z is in a Z-region and in addition $|z| = \rho_0, \rho_1, \rho_2, \cdots$.
In both cases the F of (61) is either unity or an E-function of order and
degree $H \leqq N$ and has in $\mathscr{S}$ the special structure (57), H replacing N.

Proof. When $N = 0$, Theorem III is true and trivial, for there is only
one $\mathscr{S}$-region, the entire z-plane exterior to the C_0-circle, and it is a zero-free
W-region. The E in this case reduces to a single term T; $F \equiv 1$; $\eta \equiv 0$;
$w = 3/4$; $b = 1/2$.

Granting that Theorem III applies to all E's of order $H < N$, it remains
to be shown that the theorem also holds for E's of order N. For this purpose
select from sequence (59) a particular region p_m. Then in accordance with
(46), (47), (53), and (54) in this p_m

(62) $E = t_m(f_m + \eta_m)$; $|\eta_m| \leqq |t'_m/t_m| < b_m < 1$

where t'_m is the sum of the absolute values of the terms of E not in the
dominant portion $\Psi_m = t_m f_m$ of E in p_m.

Also single out the function Ψ_s which corresponds to the central p_s-region
of the P_s in which p_m is located. The order H of this Ψ_s is less than N,
so that, by hypothesis, Theorem III applies to Ψ_s and the z-plane can be cut
up into a finite number of subregions, $\mathscr{S}_s$, each a W or Z-region of Ψ_s in which

(63) $\Psi_s = T_s(F_s + \eta_s)$ with $|\eta_s| \leqq |t'_s/T_s| < b_s < w_s < 1$,

where t'_s is the sum of the absolute values of the terms of Ψ_s not in the
dominant portion $\Psi_D = T_s F_s$ of Ψ_s in $\mathscr{S}_s$. The F_s is either unity or an E
of order and degree $G \leqq H$ and has in $\mathscr{S}_s$ the special structure (57), G re-
placing N. Furthermore, if $\mathscr{S}_s$ is a W-region of Ψ_s,

(64) $|F_s| > w_s$ in $\mathscr{S}_s$

and, if it is a Z-region of Ψ_s and $|z| = \rho_i$, $i = 0, 1, \cdots$, then (64) still
holds. Note, as the proof progresses, that if the ρ_i's are chosen in the manner
previously indicated, the same set of ρ_i's can be used for the Ψ's as for the E.

Certain of these $\mathscr{S}_s$ overlap the chosen p_m. Let $\mathscr{S}$ denote the portion
of the z-plane common to p_m and a particular one of these overlapping $\mathscr{S}_s$-
regions. Such an $\mathscr{S}$ is by definition either a Z or W-region of E depending
upon whether $\mathscr{S}_s$ is a Z or W-region of Ψ_s.

When the D-diagrams are re-examined to select the dominant terms of E
in $\mathscr{S}$, we discover that each term of Ψ_m is also a term of Ψ_s. This, coupled
with (62), implies that no term of E, or Ψ_s, can be dominant in $\mathscr{S}$ unless

it is also a term of Ψ_m. On the other hand (63) implies that no term of Ψ_m, or Ψ_s, is a dominant term in δ unless it is also a term of Ψ_D. The dominant terms of both Ψ_s and E must, therefore, be those common to Ψ_m and Ψ_D. There is, of necessity, at least one such common term, for a hypothesis to the contrary would lead at once to the absurd conclusion that $| T_s/t'_m |$ is both greater and less than unity; see (62) and (63).

Let Ψ represent the sum of the dominant terms, i. e. those common to Ψ_m and Ψ_D and let T be any particular one of them. Place $F = \Psi/T$. Make $T_s = t_m = T$. Then the terms of (ηT) in (61) fall at once into two categories: those in E, but not in Ψ_m, and those in Ψ_m, but not in Ψ_D. The sum of the absolute values of the terms in the first category divided by T does not exceed b_m, see (62); and the sum of the absolute values of the terms in the second divided by T does not exceed b_s, see (63). Hence in (61)

$$| \eta | \leqq | t'/T | < b_m + b_s = b_0.$$

To make $b_0 < 1$, ρ_0 is chosen large enough at the outset to keep $b_m < (w_s - b_s)/2$. Such a choice of ρ_0 obviously can be made if $m = 0, 1, \cdots$ or N; see (46); but, if $m = N + 1$, not only ρ_0, but also Δ_N, must be sufficiently large. Therefore requirement $\Re_N$ demands that Δ_N, as well as ρ_0, be chosen large enough to keep $b_m < (w_s - b_s)/2$. We presume that Δ_N and ρ_0 have been so chosen.

All terms of $T(F_s - F)$ are in Ψ_D and are not in Ψ; therefore they are also not in Ψ_m. Hence by (62) and (64)

$$| F | = | F_s - (F_s - F) | > w_s - b_s = w_0 > b_0$$

in δ if δ is a W-region, or if δ is a Z-region and $| z | = \rho_0, \rho_1, \cdots$. Clearly then (61) holds if $w = w_0$, $b = b_0$, and z is in an δ-region of a P_s.

As to the structure of F, note first that if F_s is unity, so also is F. If F is not unity, it is the sum of certain terms picked from F_s and, by hypothesis, F_s is an E of degree and order $G \leqq H < N$ and has in δ the special structure exhibited in (57), G replacing N. The deletion of terms from F_s does not increase the order, nor does it destroy the special structure (57). Therefore Theorem III holds for δ-regions in P_s-sectors of N-th ordered E's. It is also obvious that, if the appropriate terms are cast out of E to leave Ψ_s, then in δ

(65) $$\Psi_s = T(F + \eta_s) \quad \text{and} \quad | \eta_s | < b < w < | F(z) |$$

if δ is a W-region, or if δ is a Z-region and $| z | = \rho_0, \rho_1, \cdots$.

To complete the proof of Theorem III, Z_N and W_N's have yet to be considered. Each such region, itself, serves as an δ-region, for it is clear from (42) and (44) that, in either a Z_N or W_N, E takes the form

$$E = T_N(F_N + \eta) \quad \text{with} \quad |\eta| < |T'_N/T_N| < 1/\rho^B.$$

Furthermore, before a W_N can be marked off, a $w > 0$ must be found as explained in **7** and then the boundaries of W_N adjusted so that $w/2$ is less than $|F_N(z)|$ in W_N. Hence if ρ_0 is taken sufficiently large, which we suppose, there is a constant b such that $|T'_1/T_1| < b < w/2 < |F_N(z)|$ in W_N. The situation in a Z_N is entirely analogous, for Corollary II asserts that in such a region, there exists a $w < |F_N(z)|$ when $|z| = \rho_0, \rho_1, \cdots$. These facts complete the proof of Theorem III.

9. The variation in amplitude of E on C_i-circles. To begin the study of the zero-count for E, lump all adjoining Z's into single larger O-bands. When a Z is bounded on both the left and right by W-regions, it in itself serves as an O. These O's are separated from the W's by ξ-curves of type (30). The boundary curves are considered part of the W-regions. Theorem III then makes it clear that E will not vanish on either the boundary curves or on the C_i-circles.

LEMMA L. 3. *When z crosses a g_{N+1}-strip on any C_i, or only goes part way across on C_i, $i = 0, 1, \cdots$, the $|V.A.E_{-1}| < B$.*

Here, as in subsequent inequalities, B is a fixed upper bound independent of ρ_i, and the phrase " variation in amplitude of E_{-1} " is abbreviated by writing $V.A.E_{-1}$.

In order to substantiate L. 3, refer to Theorem III and note that in a Z_N $E_{-1} = T(F + \eta)$, $|\eta| < b$; and that on C_i $|F| > w > b$. Let a_i denote the portion of the circle C_i covered by Z_N; then when z runs along the arc a_i the

$$V.A.E_{-1} = V.A.T + V.A.(1 + \eta/F) + V.A.F.$$

To compute the $V.A.T$, use (35); insert the proper value of j; set $\rho = \rho_i$, and let σ vary over the appropriate part of the range $(-\Delta_N, \Delta_N)$. The only terms of (35) which will then vary are B_{N+1} and $\epsilon(\rho)$ and their fluctuations will be bounded. Moreover the bound is independent of ρ_i. Hence on a_i the $|V.A.T| < B$. Since $|\eta/F| < 1$ on a_i, the $|V.A.(1 + \eta/F)| < \pi$.

To estimate the $V.A.F$, write F in form (57); then replace ρ by ρ_i and erase the $\epsilon(\rho_i, \sigma)$'s to convert F into an expression of the form

$$F^*_i(\sigma) = \sum_{j=1}^{J} c_j \exp(\lambda_j \sigma)$$

where the λ's are real, and the c's, complex constants, known and fixed, the c values being dependent upon ρ_i. J is independent of ρ_i. Although the

$\epsilon(\rho_i, \sigma)$'s have been discarded, yet F^*_i approaches uniformly F on a_i as $\rho_i \to \infty$. From this, L. 1, and (61) we infer that the corresponding amplitudes of F^*_i and F are nearly equal and hence that as z runs over a_i

$$(66) \qquad\qquad | V.A.F - V.A.F^*_i | < B.$$

The $| V.A.F^*_i | < \pi J$, for neither the real nor the imaginary part of $F^*_i(\sigma)$ can vanish more than J times, see Pólya and Szegö [6]. As a consequence of this and (66), the $V.A.F$ is bounded and the bound is independent of ρ_i. The same conclusion is reached with no change in reasoning if a W_N instead of a Z_N is used; hence L. 3 is correct.

LEMMA L. 4. *Given an E-function, a P_s-sector of E, and the corresponding Ψ_s. As z travels along the C_i-circles in P_s or along any continuous curve in a W-region in P_s the $| V.A.E - V.A.\Psi_s | < B.$*

This lemma is an immediate consequence of (61) and (65).

LEMMA L. 5. *Given a term $T = P \exp Q$ of E_{-1}, if z runs along C_i from the curve*

$$(67) \qquad\qquad \xi_R: \quad \theta = \Theta_{R-1} + (m_R \log \rho + m_{R+1})/\rho^R$$

on the right to the curve

$$(68) \quad \xi_L: \quad \theta = m_0 + n_1/\rho + \cdots + n_{L-1}/\rho^{L-1} + (n_L \log \rho + n_{L+1})/\rho^L$$

on the left and if $R = L = N$, then

$$(69) \quad | V.A.T - \mathcal{P}_{N-1}(\rho_i) - NA_0(n_L - m_R) \log \rho | < B, \quad (i = 0, 1, \cdots),$$

where $\mathcal{P}_{N-1}(\rho_i)$ is a polynomial of a degree not exceeding $N - 1$ with coefficients independent of ρ_i.

If both R and $L > N$, both n_L and m_R in (69) should be replaced by zeros. If $R = N$, but $L > N$, replace only n_L by zero. Similarly, if $L = N$, but $R > N$, replace only m_R by zero.

When the $V.A.T$ is computed by means of (35), (69) is verified at once if $R = L = N$. But to use (35), when R exceeds N, and $L = N$ the equation for ξ_R should be rewritten in the form $\theta = \Theta_{N-1} + M_{N+1}/\rho^N$ where

$$M_{N+1} = m_N + m_{N+1}/\rho + \cdots + m_{R-1}/\rho^{R-N-1} + (m_R \log \rho + m_{R+1})/\rho^{R-N}$$
$$= m_N + o(1).$$

By this artifice (35) can be used and L. 5 immediately verified even when R, or L, or both exceed N.

10.   Variation in amplitude on radial ξ-curves.

COROLLARY III. *Given an E-function F_N of structure* (57) *which is uniformly bounded from zero on a curve* (55). *Then the amplitude $\Phi(\rho)$ of F_N defined on* (55) *as a continuous function of ρ is of the form*

$$\Phi(\rho) = d_N\rho^N + d_{N-1}\rho^{N-1} + \cdots + d_1\rho + O(1),$$

d's real and constant.

Proof. Since $F_N(\rho, \sigma)$ approximates a function of type (13) both in modulus and amplitude more and more accurately as ρ increases, Corollary III follows at once from Theorem II and L. 1.

LEMMA L. 6. *As z traverses a ξ-curve separating a Z_N-strip from a W-region and runs from C_0 to C_i the*

$$\left| V.A.E_{-1} - \mathcal{P}_N(\rho_i) - Nm_NA_0 \log \rho_i \right| < B.$$

Proof. Since the equation of the radial boundary ξ between Z_N and W_N is given by (55), σ fixed, ρ varying, and since ξ is considered a part of W_N, (61) is applicable and shows that as z travels on ξ, the

$$\left| V.A.E_{-1} - V.A.T - V.A.F \right| < \pi.$$

It is clear from (35) that

(70) $$\left| V.A.T - B_0\rho_i^N - \cdots - B_{N-1}\rho_i - Nm_NA_0 \log \rho_i \right| < B$$

as ρ runs from ρ_0 to ρ_i. For the same range of ρ values, Corollary III shows that

(71) $$\left| V.A.F - d_N\rho_i^N - \cdots - d_1\rho_i \right| < B.$$

These facts substantiate L. 6 when ξ separates a Z_N from a W_N, but to treat the case when ξ separates a Z_N from a P_s-sector, consider ξ a part of P_s. Then by L. 4 on ξ

(72) $$\left| V.A.E_{-1} - V.A.\Psi_s \right| < B.$$

Thus the $V.A.\Psi_s$, except for a certain bounded correction, is the same as the $V.A.E_{-1}$. Let the order of Ψ_s be H, then $H < N$. The particular W-region of Ψ_s in which ξ is located either is, or is not, a W_H-region of Ψ_s. If it is, W_H is bounded on the left by a curve ξ_L of type (55), H replacing N, σ fixed. By Theorem III $\Psi_s = T(F + \eta)$ in W_H and $|\eta| < w < |F|$. This F is an E of order and degree H and it has in W_H the structure (57), H replacing N. The $V.A.T$ on ξ as ρ varies from ρ_0 to ρ_i is again given by (70); but to estimate the accompanying $V.A.F$ on ξ, let z start at the intersection

of ξ and C_0, travel counter-clockwise along C_0 until reaching ξ_L, then run radially out along ξ_L to C_i and then back along C_i in a clockwise direction to ξ. Such a transit for z causes the same $V.A.F$ as going directly over ξ from C_0 to C_i for F does not vanish in W_H. The $V.A.F$ on ξ_L, according to Corollary III, is given by (71), H replacing N. Also there is an upper bound B on the $|V.A.F|$ as z runs from ξ to ξ_L, or vice versa, on either C_0 or C_i, see L. 3.

These facts substantiate L. 6 when ξ is located in a W_H-region of Ψ_s. If ξ is not located in such a region, it must be located in a P_s-sector of Ψ_s and our reasoning can be repeated. At each new stage in the reasoning attention is directed to an E of an order lower than any previously considered. Either the $V.A.$ of this new function on ξ is determined and L. 6 verified, or in the extreme case an E of zeroth order is reached. Such a function consists of but a single term T and its $V.A.$ on ξ is given at once by (70). Thus in all cases L. 6 is correct.

11. An estimate of the number of zeros in an O-band.

LEMMA L. 7. *As z runs along an arc A of circle C_i in an O-band from the curve ξ_R, see (67), to the curve ξ_L, see (68)*

$$(73)\quad |V.A.E_{-1} - \mathscr{P}_{N-1}(\rho_i)| < B \text{ if } R \text{ and } L > N;\quad (i = 0, 1, \cdots).$$

Proof. If $N = 1$, ξ_R and ξ_L are necessarily located in the same g_2 and L. 7 degenerates into L. 3. Suppose, therefore, that $N > 1$ and that L. 7 is valid for all E-functions of order less than N. When $N > 1$ the arc A may cross several Z_N-strips of E_{-1}, as well as the intervening P_s-sectors. As z moves entirely, or only part way, across each of the Z_N,

$$(74)\qquad\qquad |V.A.E_{-1}| < B \text{ by } L. 3.$$

To get the $V.A.E_{-1}$ as z crosses the intervening P_s-sectors, select a particular P_s and one of the terms t_s of the corresponding Ψ_s. Let $\Psi_s = t_s f_s$. Then according to L. 4 as z crosses P_s on C_i

$$(75)\qquad\qquad |V.A.E_{-1} - V.A.t_s - V.A.f_s| < B.$$

By the hypothesis for the induction

$$(76)\qquad\qquad |V.A.f_s - \mathscr{P}_{H-1}(\rho_i)| < B,\qquad\qquad (i = 0, 1.\cdots),$$

for f_s is an E of order and degree $H < N$ and the same set of ρ_i's serve for both E_{-1} and Ψ_s.

The $V.A.$'s of the various t_s have yet to be added to the typical contributions in (74) and (76) to get the total $V.A.E_{-1}$. To compute these

$V.A.$'s, suppose that ξ_R is located in a Z_N, that ξ_L is in a P_s and that, on traveling to the left along A from ξ_R to ξ_L, the following $2S-1$ boundaries of g_{N+1}-strips are crossed in the order listed below:

$$\xi_{1L}: \ \theta = m_0 + m_{11}/\rho + \cdots + m_{N-1,1}/\rho^{N-1} + (m_{N1}\log\rho + \Delta_N)/\rho^N;$$
$$(77) \ \ \xi_{jR}: \ \theta = m_0 + m_{1j}/\rho + \cdots + m_{N-1,j}/\rho^{N-1} + m_{Nj}\log\rho/\rho^N - \Delta_N/\rho^N;$$
$$(j = 2, 3, \cdots, S)$$
$$\xi_{jL}: \ \theta = m_0 + m_{1j}/\rho + \cdots + m_{N-1,j}/\rho^{N-1} + m_{Nj}\log\rho/\rho^N + \Delta_N/\rho^N.$$

As z travels from $\xi_{j-1,L}$ to $\xi_{j,R}$, it crosses an intervening P_s and L. 5 states that the $V.A.$ of the corresponding t_s differs from

$$(78) \qquad\qquad \mathcal{P}_{N-1}(\rho_i) + NA_{s0}(m_{Nj} - m_{N,j-1})\log\rho_i$$

by not more than a B.

Similarly as z travels from ξ_{SL} to ξ_L the $V.A.$ of the appropriate t_s differs from

$$(79) \qquad\qquad \mathcal{P}_{N-1}(\rho_i) - 2NA_{s0}m_{NS}\log\rho_i$$

by not more than a B. Totalling these results it is found that the contributions of the t_s terms to the $V.A.E_{-1}$ differs from

$$(80) \qquad\qquad \mathcal{P}_{N-1}(\rho_i) - NA_0 m_{N1}\log\rho_i$$

by less than a B. One log term is indicated; the others have cancelled out. This cancelling takes place because the various A_{s0}'s of (78) and (79) are all equal to the same constant A_0, corresponding as they do to terms t_s which are all associated with points on the same side of the D_0-diagram. Result (80) gives the impression that a log term should have been added to (73), but by hypothesis the entire ξ_R-curve, is located in a g_{N+1}-strip bounded on the left by ξ_{1L}. This means that the m_{N1} of (80) is zero, for otherwise ξ_R would eventually run outside g_{N+1} as $\rho \to \infty$. L. 7 is therefore correct for the given configuration.

There are, however, other possible configurations; ξ_L might be in a g_{N+1} and ξ_R in a P_s, ξ_R and ξ_L might be either in the same or different g_{N+1}'s, or as a final possibility ξ_R and ξ_L might be in the same or different P_s-sectors. All these configurations, as well as the one just considered, can be analyzed in much the same way and in each case L. 7 is verified. The details need not be repeated.

LEMMA L. 8. *Given a curve ξ_L of type* (68), $L > N$, *running out to infinity in an O-band of E_{-1}. If z starts on the C_0-circle where ξ_L intersects C_0 and travels to the right (left) along C_0 until arriving at the right (left)*

radial boundary ξ_R of O, then runs out radially along ξ_R as far as C_i and then back to the left (right) along C_i until ξ_L is reached, then

$$| V.A.E_{-1} - \mathcal{P}_N(\rho_i) | < B, \qquad\qquad (i = 0, 1, \cdots).$$

Proof. When $N = 1$, the O-band is necessarily a Z_1-region bounded on the right by either a W_1-region or a P_s-sector. If it is a P_s, on ξ_R E_{-1} $= T(1 + \eta)$ with $|\eta| < 1/2$, and if it is a W_1, $E_{-1} = T(F_1 + \eta)$ with $|\eta| < w < |F_1|$. This last equation also holds on C_0 and C_i in Z_1, see Theorem III.

From these facts, (35), and Corollary III, it is evident that $V.A.E_{-1}$ can not differ from $\mathcal{P}_1(\rho_i) + m_1 A_0 \log \rho_i$ by as much as a B when z runs from C_0 to C_i on ξ_R. Moreover m_1 is zero, for ξ_L remains in Z_1 as $\rho \to \infty$. On crossing C_i or C_0 from ξ_R to ξ_L, or vice versa, $|V.A.E_{-1}| < B$ by L. 3. Hence when $N = 1$, L. 8 is correct.

Next suppose that L. 8 is valid for all E's of order less than N. Let E_{-1} be of order $N > 1$. In this more complicated case the portion of the O to the right of ξ_L will consist in general of several Z_N-strips and intervening P_s-sectors. To be specific suppose that, when

$$\xi_{1R}: \quad \theta = m_0 + m_{11}/\rho + \cdots + m_{N-1,1}/\rho^{N-1} + (m_{N1} \log \rho - \Delta_N)/\rho^N$$

is added to (77), (77) becomes a complete list of the boundaries of the Z_N's in the O-band to the right of ξ_L. As z crosses these Z_N's on C_0 or C_i, $|V.A.E_{-1}| < B$. In the P_s-sectors between ξ_{1R} and ξ_L, (75) is again correct and so is (76). The contribution of each t_s is computed as in L. 7. When these contributions are combined, (80) is the resultant total $V.A.E_{-1}$ on C_i between ξ_{1R} and ξ_L. This time m_{N1} is not necessarily zero. On C_0 $|V.A.E_{-1}| < B$.

If ξ_{1R} happens to be the right radial boundary of the O-band, then as z runs from C_0 to C_i on ξ_{1R}

$$(81) \qquad | V.A.E_{-1} - \mathcal{P}_N(\rho_i) - N A_0 m_{N1} \log \rho_i | < B \text{ by L. 6.}$$

Combining (76), (80), and (81), L. 8 is found to be correct for the given configuration.

In case ξ_{1R} is not the right radial boundary of the O, the part of the O-band which extends to the right of ξ_{1R} will then be located within a single P_s. In this P_s (75) is applicable. As z runs directly over ξ_{1R} from C_0 to C_i

$$| V.A.t_s - \mathcal{P}_N(\rho_i) - N A_0 m_{N1} \log \rho_i | < B.$$

This same $V.A.$ is produced by letting z start at the intersection of C_0 and ξ_{1R},

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travel along C_0 until reaching the right radical boundary of O, then run out this boundary to C_i and finally come back along C_i to ξ_{1R}. According to the hypothesis for the induction, the $V.A.i_s$ over this same round about path differs from a $\mathcal{P}_H(\rho_i)$, $H < N$, by less than a B and again L. 8 is correct. Other configurations can be analyzed in the same fashion and in all cases L. 8 is verified.

Enough facts have been assembled to prove our final theorem. One detail is lacking; an order should be assigned to each O-band. In Theorem III it is stated that $E = T(F + \eta)$ in an $\mathcal{S}$-region. If the order of this F is H, let H be the order of $\mathcal{S}$. Since $\mathcal{S}$ is a Z (or W) region, let H be the order also of the Z (or W) region. Finally let the order of an O-band be that of the highest ordered Z-region contained within the O. A subscript attached to a W, Z, or O will indicate the order.

THEOREM IV. *In an O_M-band of an E-function of order $N \geqq M$,*

$$(82) \qquad\qquad | \mathcal{N}(\rho_i) - \mathcal{P}_M(\rho_i) | < B, \qquad\qquad (i = 0, 1, 2, \cdots).$$

Proof. Without loss of generality assume both the order and degree of E to be N. Consider first an O_N-band. Let the boundaries of the various Z_N's located in O_N be ξ_{1R} and the curves listed in (77). Discard the portion of the O_N-band beyond C_i and then let z traverse once counter-clockwise the perimeter of the remaining fragment of O_N which runs from C_0 to C_i. When once the circuit is made, the total $V.A.E/2\pi$ gives the zero count for the fragment.

To be specific let z start at the point P, the intersection of ξ_{8L} and C_0, and travel along C_0 clockwise until reaching ξ_{1R} at the point Q. Thus far the $| V.A.E | < B$. If ξ_{1R} is flanked on the right by a W-region, let z run out along ξ_{1R} to point R, the intersection of ξ_{1R} and C_i. According to L. 6 this causes a $V.A.E$ which differs from $\mathcal{P}_N(\rho_i) + N m_{N1} A_0 \log \rho_i$ by less than a B. On the other hand, if ξ_{1R} is flanked on the right by Z-strips of order less than N, z must skirt these strips on the right to get to R. In order to do this let z first run along C_0 until it reaches the right radial boundary of O_N; then go along this radial boundary to C_i and finally run back to the left along C_i to R. While z skirts these Z-strips of order less than N, and runs from Q to R, as described, the path followed remains in a single P_s-sector. Therefore L. 4 is applicable and implies that if t_s is a term of Ψ_s and $f_s = \Psi_s/t_s$, then

$$(83) \qquad\qquad | V.A.E - V.A.f_s - V.A.t_s | < B.$$

This f_s is an E of degree and order $H < N$. L. 8 is applicable and states that as z runs from Q to R and skirts the O-band the $| V.A.f_s - \mathcal{P}_H(\rho_i) | < B$.

To get the corresponding $V.A.t_s$, z need not go on the round about path skirting Z-strips of lower order, but may go directly over ξ_{1R} from Q to R; hence (35) can be used and shows that

$$| V.A.t_s - \mathcal{P}_N(\rho_i) - Nm_{N1}A_0 \log \rho_i | < B.$$

Once at R, z travels along C_i, counter-clockwise until reaching the point S, the intersection of ξ_{SL} and C_i. In getting from R to S a finite number of Z_N-strips, or portions thereof, are crossed and while z crosses these Z_N's, according to L. 3, $| V.A.E_{-1} | < B$. In getting from one of these Z_N's to the next on a C_i, z crosses a P_s. For such a crossing (83) and L. 7 are applicable and therefore the corresponding $| V.A.f_s - \mathcal{P}_{H-1}(\rho_i)| < B$. Utilizing L. 5 and lists like (77) the contributions over the respective P_s-regions of the corresponding t_s terms are computed and added together. We find that the total contribution of these terms to the $V.A.$ differ from

$$\mathcal{P}_{N-1}(\rho_i) + N(m_{NS} - m_{N1})A_0 \log \rho_i \quad .$$

by less than a B.

In getting from S back to P and completing the circuit the details are similar to those in getting from Q to R and need not be repeated. The contribution to $V.A.E$ over this last portion of the circuit differs from $\mathcal{P}_N(\rho_i) - Nm_{NS}A_0 \log \rho_i$ by less than a B. Totaling the changes in amplitude enumerated over the different parts of the circuit, the logs cancel and (82) is verified, at least when $M = N$. Theorem IV must therefore be true when $N = 1$ for in this case there are no O-bands other than the O_1's.

Next suppose that Theorem IV has been established for all E's of order less than N. Select a particular O_M, $M < N$. This band will necessarily be located in a single P_s. Hence by L. 4 on the radial boundaries of O_M, as well as on the arcs of the C_i's in P_s, the $| V.A.E - V.A.\Psi_s | < B$. Since the order H of Ψ_s is less than N, Theorem IV applies to Ψ_s by hypothesis and therefore $| V.A.\Psi_s - \mathcal{P}_M(\rho_i)| < B$; $M \leqq H < N$; and as a consequence $| V.A.E - \mathcal{P}_M(\rho_i)| < B$. This completes the demonstration of Theorem IV.

It is evident from (82) and the relative dense distribution of the C_i's that, if $\rho_{i+1} = \rho_i + d_i$, the difference $\mathcal{P}_M(\rho_i + d_i) - \mathcal{P}_M(\rho_i)$ is a polynomial of at most degree $M - 1$ in ρ_i with coefficients dependent upon d_i, but less in absolute value than some fixed bound B independent of ρ_i. Hence the number of zeros in an O_M-band between C_i and C_{i+1} is a quantity of the order ρ_i^{M-1}. Therefore

$$\mathcal{N}(\rho) = k\rho^M\{1 + O(1/\rho)\}.$$

Once the band is chosen, k is fixed. This formula gives more freedom in the choice of ρ than (82), but less precision in the zero count.

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FIXED ELEMENTS AND PERIODIC TYPES FOR HOMEO-MORPHISMS ON S. L. C. CONTINUA.[*]

By G. E. Schweigert.

The results to be established in this paper consist of eight characterizations for the elementwise periodic type homeomorphism, certain other characterizations related to these, and some contributions toward classifying the general homeomorphisms $T(S) = S$. In all cases the space S is a semi-locally-connected continuum. It is hoped that these characterizations not only give a many sided view of the concepts they represent, but are sufficiently different so that, compared part by part, they show something of how certain finite and infinite orbits intermingle in the general homeomorphism. · On the other hand one of these is developed especially for dealing with orbit decompositions, and is applied in the fifth section to give a theorem of this type. Where periodicity properties enter there is a close relationship between the present work and that of Whyburn and Ayres; in fact the methods of this paper are sometimes chosen so as to foster this relation. However, when proofs which follow the same lines as those given before may be used, they are not repeated here. In this connection it is to be noted that the homeomorphisms studied here include the pointwise almost periodic homeomorphisms of the earlier work. The second section of this paper, wherein periodicity is not assumed, establishes new results concerning fixed elements and fixed points. These propositions, together with the well known fixed element theorem of Ayres, are then used liberally throughout this paper.

It is assumed that the reader is familiar with the cyclic element theory [1] for semi-locally connected continua. In addition to the concepts associated with this theory we use here the particular term *bordered* A-set to mean that, for a given A-set B, there exists a point x and a true cyclic element E such that $E \cdot B = x$. Thus we may speak of either bordered or unbordered A-sets. Three considerations of a structural nature are used symbolically here. The boundary, $F(G) = \overline{G} - G$, of an open set, G; and the interior (greatest open

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[1] See G. T. Whyburn, *Analytic Topology*, Colloquium Publication (1942), pp. 64-98. We refer to this book hereafter as ATW with the numbers for theorems in brackets.

set) of an arbitrary set X, denoted by Int X. The letter L is consistently used to denote the set of all end points of a space S.

Various items of an analytic nature are needed. We shall be concerned with a homeomorphism $T(S) = S$ of a semi-locally-connected continuum S onto itself. Hence, as is customary, we set $T(x) = T^1(x)$, $TT(x) = T^2(x)$, and so on, whence $T^n(x)$ exists and is said to be a power of $T(x)$. To proceed, using these powers, a given set B for which $T^n(B) = B$ is said to be invariant without mentioning T. The term fixed set, as used here, does not necessarily mean a set of fixed points. If n is the least positive integer such that $T^n(B) = B$ then n is said to be the *period of B under T*; and the set $B + T(B) + T^2(B) + \cdots + T^{n-1}(B)$ is the *orbit of B under T*. The symbol $T^n(X)$ denotes the image of X under the identity homeomorphism, when $n = 0$, and the image of X under the inverse of T, when $n = -1$. The sum of all sets $T^n(X)$, where the range for n includes *zero* and the negative integers, is the *infinite orbit of X under T* provided these image sets are distinct. We are interested in a special type of homeomorphism for which certain finite orbits are known to exist. More specifically, T is said to be *elementwise periodic on a cyclic element E* in S provided there exists an integer n such that $T^n(E) = E$. Thus we may speak of a homeomorphism which is elementwise periodic on all cut points and true cyclic elements of S; in other words a homeomorphism T which is *elementwise periodic on all cyclic elements E of S that lie in S — L*. It is to be noted that if $W = T^n$ then the homeomorphism W is also elementwise periodic on $S — L$.

We conclude this section by giving some theorems which are fundamental as a basis for this paper. These theorems are due to Ayres and Whyburn and are to be found in § 4, Chapter XII, of *Analytic Topology*, by G. T. Whyburn. They were established under the assumption that T is pointwise almost periodic, but the proofs as presented in *Analytic Topology* allow a change in hypothesis that is essential to the needs of this paper. In each case we assume instead that T has the fixed point property defined below. This property holds [2] if T is pointwise almost periodic, but the converse is not true, hence we have a slight generalization for each theorem. The method of proof remains the same.

(1.1) DEFINITION. If, for every division $S = H + K$ of S into continua H and K such that $H \cdot K = p \, \epsilon \, S$ and $H \cdot T(H) \neq 0 \neq K \cdot T(K)$, it follows that $T(p) = p$, then T is said to have the *fixed point property for nodal sets*.

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[2] ATW [4.21] p. 247.

(1.2)   LEMMA [3] [4. 22W].   *If a node N of S is invariant and $N \neq S$, there exists a fixed cut point of S.*

(1.3)   THEOREM [4. 3A].   *If the cyclic elements C_1 and C_2 are invariant, every cyclic element of the chain $C(C_1, C_2)$ is invariant.*

(1.4)   THEOREM [4. 4A].   *The sum I_T of all invariant cyclic elements of S is a non-empty A-set.*

(1.5)   THEOREM [4. 5A].   *If C_1 and C_2 are distinct cyclic elements of S and $T(C_1) = C_2$, the chain $C(C_1, C_2)$ has one and only one cyclic element which is invariant.*

2. Fixed elements, fixed points; subsequent characterizations.   Throughout this paper $T(S) = S$ denotes an arbitrary homeomorphism of a semi-locally-connected continuum S onto itself.   We do not assume periodicity properties in this section and because of this generality the first three theorems may eventually be put to better service than that shown here.   However, we do use, in the final characterization, certain properties from the theorems above.   Thus a distant contact with periodicity is maintained later in this section.

(2.1)   THEOREM.   *If $N \neq S$ is an invariant node then there exists in S another invariant cyclic element $E \neq N$.*

Proof. If N is a true cyclic element it contains one cut point x which is fixed. Let $E = x$ and the theorem is true. We may therefore assume that $N = y$ is an end point of S. Let $M \neq N$ be a node and consider the cyclic chain $Y = C(M, y)$. Since $T(M)$ is a node, either $T(M) = M$ establishes the theorem, or (when $T(M) \neq M$) we get a cyclic element $K \subset Y \cdot T(Y)$ such that the cyclic chain $C(K, y) = Y \cdot T(Y)$. Furthermore $M \neq K \neq y$, since y is an end point. As the first of two cases, we assume $T(K) \neq K$ and $T(K) \subset C(K, T(M)) \subset T(Y)$. It follows that the cyclic chain $X = C(K, y)$ is a proper subset of its image and this leads to the infinite strictly monotone sequence $X \subset T(X) \subset T^2(X) \subset \cdots \subset T^n(X) \subset \cdots$. We wish to show that the end elements $T^n(K)$ converge to a point x. This will occur if (and only if) $H = \Sigma T^n(X)$ has for its closure a cyclic chain. In this connection we observe that H is connected and is of a special nature in that it is a sum of nested chains. Briefly the steps showing that $\bar{H}$ is a cyclic chain are as follows: (1) H is an H-set, hence $\bar{H}$ is an A-set and each point of $\bar{H} - H$

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[3] For convenience we include in the text [4. 22 W] meaning ATW [4. 22] p. 247. The letters W and A refer to Whyburn and Ayres.

is either a cut point or an end point of S; (2) $\bar{H} - H$ contains only end points of $\bar{H}$; (3) $\bar{H} - H$ contains only one end point x of H because of the nested chain development of H. Thus we have $\operatorname{Lim} T^n(K) = x \,\epsilon\, S$. The points x and y are distinct since they lie at opposite ends of the chain $\bar{H}$, and the property $T(R) \overset{\ast}{\subset} R$, for $R = \Sigma T^n(K)$, yields the result that x is a fixed point. Having completed our discussion of this case we assume now that $T(K) \neq K$ and $T(E) \subset C(K, y)$. Here X contains $T(X)$ as a proper subset and we may repeat the argument above using T^{-1} instead of T. This completes the proof of the theorem.

(2.11)   COROLLARY.   *If S is a dendrite and $N = y$ is a fixed end point, then $E = x$ is a fixed point distinct from y.*

Under suitable conditions our theorem may be extended to include homeomorphisms for which $T(S)$ is a proper subset of S. In the next corollary the set M may be the finite orbit of some cyclic element E, or M may be the infinite orbit of E provided the sets $T^n(E)$ exist and are distinct for positive and negative integral values of n. Particular examples of many types can be found.

(2.12)   COROLLARY.   *If $T(S) \subset S$, $N \neq S$ is an invariant node of S, and M is an invariant set in S such that M contains a point of $S - N$, then there exists a cyclic element E of S such that $E \neq N$ and $T(E) = E$.*

Proof.   The least A-set A which contains the invariant set $M + N$ is likewise invariant under T. Since $A \neq N$ we apply the theorem to the space A.

(2.2)   DEFINITION.   A division $S = H + K$ of S into continua H and K such that: (1) $H \cdot K = p$, (2) p is a cut point of S, (3) $H - p$ is a component of $S - p$, (4) H contains a cyclic element E of S for which $T(E) = E$, is called a *special decomposition* of S and is denoted by $S(p)$.

There is at least one special decomposition $S(p)$ relative to each given cut point p of S. When p is fixed we may put $H = \bar{M}$ for any component M of $S - p$. Otherwise there exists an M which contains a fixed cyclic element.[4] These decompositions are used to study the orbit of p and certain sets in its complement. We take up the investigation in the next paragraph, and it culminates in two theorems stated later. These theorems are of prime importance to this paper.

We suppose that some special decomposition $S(p) = H + K$ is given and fixed. It is also assumed that $T^n(K) \cdot K \neq 0$, where n is the least positive

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[4] W. L. Ayres, *Fundamenta Mathematicae*, vol. 16 (1930), pp. 332-336. Also ATW [2.51] p. 242.

integer giving this intersection. Since $T^n(p) = p$ is useful in establishing the fixed point property for nodal sets we attempt to classify all other possibilities; that is, until the end of this discussion we shall be concerned with the case $T^n(p) \neq p$. It will be demonstrated first that of the two sets H and $T^n(H)$ one is always a proper subset of the other. Then certain consequences will be stated and finally the assumption $T^n(K) \cdot K \neq 0$ will be removed.

We intend to show later that the orbit of p is infinite, hence we note now that $T^k(p) \neq p$, for $k < n$, since $T^k(K) \cdot K = 0$. We also need $H - p \neq 0$ and $K - p \neq 0$; these are immediate since $H = p$ and the fixed element in $T^n(H) \cdot H \neq 0$ imply $T^n(p) = p$. To proceed with the main investigation, assume $T^n(p) \in H - p$ and also suppose that $T^n(H - p)$ is not contained in $H - p$. In view of the fact that $H - p$ is a component we may say that $H - p$ is open and $T^n(H - p)$ is connected. But the connected set $T^n(H - p)$ must contain $p = F(H - p)$ since it meets the open set $H - p$ (fixed element) and its complement K (by assumption). We now repeat this argument using K as the connected set and $T^n(H - p)$ as the open set, with K meeting $T^n(H - p)$ at p and its complement $C(T^n(H - p)) = T^n(K)$, to get $F(T^n(H - p)) = T^n(p) \in K$ contrary to the assumption. This contradiction allows $T^n(H - p) \subset H - p$ which in turn gives $T^n(H) \subset H - p$ since $T^n(p) \in H - p$. Hence $T^n(H)$ is a proper subset of H. The other assumption $T^n(p) \in K - p$ gives H a proper subset of $T^n(H)$ by a similar (but shorter) argument. We may now treat both proper subset inclusions as one by basing the discussion on $f(H)$ a proper subset of H, where f denotes either T^n or T^{-n} according to the position of $T^n(p)$. In passing we note that the T-orbit of p is infinite by virtue of the proper subset relations under iteration of f plus the remarks at the beginning of this paragraph.

If we define $P = \prod_{i=1}^{\infty} f^i(H)$, it can be shown that: (1) P is invariant under f and any subset of H with this invariance is contained in P, (2) P is a nodal set (hence an A-set), (3) P is unbordered. Concerning the first statement it is obvious from the definition of P that $f(P) \subset P$ and hence we proceed at once to show that $P \subset f(P)$. If, corresponding to $x \in P$, there is no $q \in P$ such that $T(q) = x$, then $f^{-1}(x)$ is not in $f^i(H)$ for some i. Hence $ff^{-1}(x) = x$ is not in $f^{i+1}(H)$ contrary to $x \in P$. The rest of statement (1) follows from the definition of P. As to the second statement we note first that, for each i, Int $f^i(H)$ is connected and may be written in the form $f^i(H) - f^i(p)$. Thus $y \in f^{i+1}(H)$ implies $y \in$ Int $f^i(H)$ since $f^{i+1}(H)$ is a proper subset of $f^i(H)$. In particular $x \in P$ implies $x \in$ Int $f^i(H)$ for every i. Hence if M and N are components of $S - P$ with limit points x and y in P, we can conclude that, for a fixed i, $\bar{M} = M + x$ contains the point $f^i(p)$, since

$\bar{M}$ is connected and the interior of $f^i(M)$ is an open set containing x. A similar remark holds for N and $x = y$ follows from the fact that $\bar{M} \cdot \bar{N}$ contains $f^i(p)$. Finally, P is unbordered, since any true cyclic element E, such that $E \cdot P$ is a single point, will have exactly one point z in common with each set of $f^i(H)$ which does not contain E. But then $E \subset f^i(K)$ and since $f^i(H) \cdot f^i(K) = f^i(p)$ we get $z = f^i(p)$ for all sufficiently large i. This is impossible except when p is fixed. The discussion under $T^n(K) \cdot K \neq 0$ is complete.

If on the other hand, $T^n(K) \cdot K = 0$ for all $n \neq 0$, then $T^n(p) \subset H - p$ for all $n \neq 0$. And, if $T^m(p) = T^k(p)$ for $k > m$, then $p = T^{k-m}(p) \, \epsilon \, K$; hence p has an infinite orbit. Moreover $T^n(K) \cdot K = 0$ implies $T^n(R) \cdot R = 0$, for all $n \neq 0$, where R is a component of $K - p$. We now let $Q = \prod_{n=1}^{\infty} T^n(H)$ and establish $T(Q) = Q$ by the argument used above on the set P. Since T is a homeomorphism and Q is an invariant A-set we have by elementary duality the result $S - Q = \Sigma T^n(K - p)$; thus each component M of $S - Q$ is of type $T^m(R)$, where R is a component of $K - p$. Furthermore, if $z = F(M)$ and Z is the set of all z, then Z is the orbit of p.

The entire discussion will now be summarized and extended, using the terms, B is T^n-invariant, or, a T^n-fixed set, to mean $T^n(B) = B$.

(2.3) THEOREM. *Let p be a cut point of S and let $S(p)$ be a given special decomposition into sets H and K. If $T^n(K) \cdot K \neq 0$, and $n \neq 0$ is the least integer giving this intersection, then: either p is of period n; or the period of p is infinite, and there exists in $H - p$ an unbordered T^n-invariant [5] nodal set, $P \neq 0$, which contains every T^n-invariant set in H. If $T^n(K) \cdot K = 0$ for all $n \neq 0$, then, the period of p is infinite, and there exists in H an invariant A-set, $Q \neq 0$, such that $Q \cdot \overline{S - Q} = \overline{\Sigma T^i(p)}$.*

(2.4) THEOREM. *If p is a cut point of S and there is some special decomposition $S(p)$ with $T^n(K) \cdot K \neq 0$, for some (least) integer $n \neq 0$, then; either p is of period n, or there exist two T^n-fixed points, x and z, such that the cyclic chain, $C(x, z)$, with end points x and z, is the least (T^n-invariant) A-set which contains the T^n-orbit of p.*

Proof. We assume $T^n(p) \neq p$; that is an infinite orbit, under T^n, for p. Hence we must find x and z and establish suitable properties for $C(x, z)$. Only the outlines of this work are given. The details rest largely on the properties of P, from Theorem (2.3), and the proof of Theorem (2.1). Let M

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[5] It was shown that P is invariant under either T^n or T^{-n}, that is, under f; but $f(P) = P$ implies $f^{-1}(P) = P$, hence $T^n(P) = P$ and $T^{-n}(P) = P$ hold simultaneously.

be the component of $S - P$ which contains K, and let $F(M) = z = P \cdot \overline{S - P}$ define z. It can be shown that $p + T^n(p) \subset M$ and hence, using the T^n-invariance of P, we get $T^n(M) = M$, and finally $T^n(\bar{M}) = \bar{M}$. This gives $T^n(z) = z$. Since P is an unbordered A-set, z is an end point of $\bar{M}$ to which one of the sequences, $p + T^n(p) + T^{2n}(p) + \cdots$, and $p + T^{-n}(p) + T^{-2n}(p) + \cdots$, converges. These remarks will aid in showing that z is an end point of the cyclic chain $\bar{Q}$ constructed below. Now, in order to get X a proper subset of $f(X)$, where $X = C(z, p)$, we choose f to be either T^n or T^{-n} as needed. As in Theorem (2.1) the orbit $Q = \Sigma f^i(X)$ of the cyclic chain X is used to define $\bar{Q} = Q + x$. With x so defined, and $f(x) = x$, we show that $\bar{Q}$ is also a cyclic chain; in fact $\bar{Q} = C(z, x)$. Finally it is shown that each point of $C(x, z) - (x + z)$ has an infinite orbit under f, and that x and z are the only limit points of the f-orbit of p. This covers the essentials of Theorem (2.4).

We conclude this section with a cycle of characterizations to be established by means of the theorems above. These characterizations are stated in terms of concepts from the theorems of Ayres and Whyburn (listed in the introduction). Concepts such as these possess a striking simplicity and clarity; nevertheless they are not obviously the same, and one may welcome the fact that it will no longer be necessary to establish more than one of them. It will be evident that the key to their similarity lay in the common ground [6] between Theorems (2.1) and (1.2). Likewise it will be clear that the methods of Whyburn and Ayres were well in advance of the formal conclusions they stated.

(2.5) DEFINITIONS. It will be said that T has *the first, second,* or *third property of Ayres,* or *the first property of Whyburn,* according as (a1, 2, 3), or (w1), below holds.

(a1) If C_1 and C_2 are invariant cyclic elements then every cyclic element in $C(C_1, C_2)$ is invariant.

(a2) The sum of all invariant cyclic elements of S is a non-empty A-set.

(a3) For every pair of distinct cyclic elements E_1 and E_2 such that $T(E_1) = E_2$, the chain $C(E_1, E_2)$ contains one and only one invariant cyclic element.

(w1) If X is an invariant A-set in S and N is an invariant node of X such that $N \neq X$, then there exists a fixed cut point x of X.

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[6] Theorem (1.2) is, in many respects, replaced by Theorem (2.1), but gains

(2.6) THEOREM. *In order that $T(S) = S$ have*[7] *the fixed point property for nodal sets it is necessary and sufficient that one of the following statements holds:*

(a1, 2, 3) *T has one of the properties of Ayres.*

(w1) *T has the first property of Whyburn.*

Remarks. For brevity we denote the fixed point property for nodal sets by (w2).[8] In the early stages of the cycle of proof the arguments are not given; the remarks indicate modifications in the existing methods of argument.[9]

Proof. (w2) $\rightarrow$ (a1). See the reference for Theorem (1.3) $= [4.3A]$ of the introduction. To show that any point of $E(a, b)$ is fixed cite (W2).[10] Turning to the next stage of the cycle, (a1) $\rightarrow$ (a2), we refer to the argument for Theorem (1.4) $= [4.4A]$ without changes. For the third stage, (a2) $\rightarrow$ (a3), we need only cite Theorem (2.1), instead of Theorem (1.2) $= [4.22W]$, at the (only) proper place in the proof of Theorem (1.5) $= [4.5A]$.

(a3) $\rightarrow$ (w1). If E is a cyclic element in a cyclic chain $C(C_1, C_2)$, where C_1 and C_2 are fixed cyclic elements, then either E is fixed or E has an infinite orbit. This fact is well known and will be used henceforth without formality. We follow the notation of (2.5w1) and assume that no cut point of X is fixed. It follows that $N = z$ is an end point. Furthermore, since $N = z$ is an end point, we get by Theorem (2.1) a fixed element $C \neq N$ of X. But $C = y \, \epsilon \, X$, since no cut point is fixed. Hence y is an end point and $y \neq z$. Obviously there are cut points in the invariant chain $C(y, z)$, and if x is such a point then x has an infinite orbit. Since every cyclic element, except for y and z, in $C(y, z)$ has an infinite orbit the subchain $C(x, T(x))$ contains no invariant element. Thus (a3) fails to hold.

(w1) $\rightarrow$ (w2). If p is a cut point and $S = M + N$ is a decomposition into continua such that $M \cdot N = p$ we may assume that $T(M) \cdot M \neq 0$ and $T(N) \cdot N \neq 0$ for, otherwise, nothing is required of the point p. If these

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prominence again in localized form (W1) below. Note that in either form, (1.2) or (W1), it could be stated that the fixed cut point x satisfies $p(x, N) < \epsilon$.

[7] In contrast to the earlier parts of this section, T^n, for $n > 1$, is not used. Results for T^n in terms of T^m-invariance are immediate from the next theorem.

[8] Indicating the second property of Whyburn.

[9] The proofs in ATW. The original work of Ayres is not generally available; see references to him in ATW.

[10] The essential property $K \cdot T(K) \neq 0$ needed in order to apply (W2) is readily found from the invariance of C_2.

inequalities hold and p is not fixed there is obviously a special separation $S(p)$ for S into continua H and K for which $T(K) \cdot K \neq 0$. By Theorem (2.4), for the case $n = 1$, there is a cyclic chain $C(x, y)$ for which x and y are fixed end points and such that the infinite orbit of p is contained in $C(x, y)$. It follows that every point of $C(x, y)$ other than x and y has an infinite orbit, and since x is an invariant node (an end point in particular), the absence of any fixed cut points in $C(x, y)$ contradicts (w1). We have therefore shown that if (w2) fails to hold so also does (w1). This completes the cycle needed to prove Theorem (2.6).

3. Componentwise and elementwise periodicity; two characterizations.

Since elementwise periodicity follows from pointwise almost periodicity [11] and yet is by far less restrictive, it is natural that this paper should use certain concepts implied by the p. a. p. assumption; particularly those stated in terms of large structural forms. Two examples of this occur below.

(3.1) DEFINITIONS. Let $T(S) = S$ be a homeomorphism, S a semi-locally-connected space, and A an invariant true [12] A-set. A component R of $S - A$ belongs to the class $[R(x)]$ provided it contains some cut point x of S. If each component $R(x)$ has a finite period, and if this is true for every choice of A, then T is said to have *property* (α).

If an arbitrary component R of $S - A$ has a finite period, T is said to be *componentwise periodic at* A. Extending this concept, T is said to be *componentwise periodic*, if it is componentwise periodic at A, for each true invariant A-set A in S.

(3.2) THEOREM. *Let L^* denote the set of all nodes of a semi-locally-connected continuum S and let $T(S) = S$ be an arbitrary homeomorphism. In order that T be elementwise periodic on the cyclic elements of S in $S - L^*$ it is necessary and sufficient that T have property (α) and that for every integer n, T^n has the fixed point property for nodal sets [13] $(w2 - n)$.*

Proof. We assume that T has (α) and satisfies $(w2 - n)$ and we wish to show first that if p is a cut point then $T^k(p) = p$ for some k. If K is one of the continua in a special decomposition $S(p)$ and $T(K) \cdot K = 0$ then p

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[11] For locally connected continua, an unpublished result due to Ayres. For s. l. c. continua see ATW [4.6] p. 248. Pointwise almost periodicity means $P(x, T^n(x)) < \epsilon$ for some $n = n(x, \epsilon)$.

[12] The only true A-sets which are single points are the end points and cut points.

[13] $(w2 - n)$ for *every* n means that (relative to $S(p)$), $T^k(K) \cdot K \neq 0$, with k the least positive integer, implies $T^k(p) = p$. The particular value $k = n$ is dictated by the circumstances.

has an infinite orbit and the set Q of Theorem (2.3) exists. In the light of this theorem we also get the information that each component R of $K - p$ is a component of $S - Q$ and has an infinite orbit. This is contrary to property (α): for, by virtue of the fact that either $x = T(p) \, \epsilon \, K - p$ or $x = T^{-1}(p) \, \epsilon \, K - p$ holds, we are furnished with one component $R = R(x)$ of $K - p$, with $x \, \epsilon \, R(x)$, and x a cut point; hence we have by (α) one component with a finite orbit. Since $T(K) \cdot K = 0$ cannot hold we may assume $T^k(K) \cdot K \neq 0$, with k the least such integer. This allows us to use the hypothesis $(w2 - k)$ which, in turn, gives $T^k(p) = p$. Thus each cut point p has a finite period. If E is a true cyclic element of $S - L^*$ then E contains two cut points p and q with periods k and m respectively. Therefore $T^{km}(E) = E$ follows at once and the proof of the sufficiency is complete. The necessity is readily shown using Theorem (2.3) for the $(w2 - n)$ part.

The next characterization is not only the leading result of this section but one which plays a central role later in the application for which it was designed. As a means of stating it we use the concept of a contracting approximation defined below. This definition could have been given in terms of T alone and extended by substituting T^n for T throughout.

(3.3) DEFINITION. Let A be an A-set for which $T^n(A) = A$. If for every A-set C with the property $A \subset \operatorname{Int} C$ there is a third T^n-invariant A-set B such that $A \subset \operatorname{Int} B \subset C$, then we say that A *admits a contracting T^n-approximation which preserves interiority at A.*

(3.4) THEOREM. *In order that the homeomorphism $T(S) = S$ be elementwise periodic on the cyclic elements of S in $S - L$ it is necessary and sufficient that T be componentwise periodic and that each unbordered* [14] *T^n-invariant A-set A in S admits a contracting T^n-approximation which preserves interiority at A.*

Proof. The sufficiency will follow readily if it is known that, for every n, T^n has the fixed point property for nodal sets—$(w2 - n)$. We therefore show that, for a given cut point p, and any special decomposition $S(p)$ for which $T^m(K) \cdot K \neq 0$, there is a $k \leq m$ with $T^k(p) = p$. Here k denotes the least positive integer giving $T^k(K) \cdot K \neq 0$. If the period for p is not k then, by Theorem (2.3), p has an infinite orbit, and there exists in $H - p$ an unbordered T^k-invariant nodal A-set $P \neq 0$ which contains every T^k-invariant set in H. But on the other hand H is an A-set such that $P \subset \operatorname{Int} H$, and hence, by hypothesis, there is a T^k-invariant A-set B such that $P \subset \operatorname{Int} B \subset H$. It follows that $P = B$ contrary to $P \subset \operatorname{Int} B$. This estab-

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[14] Sufficient to make A a *true-A*-set.

lishes $(w2 - n)$. Now, since componentwise periodicity implies property (z), we have, by Theorem (3.2), the result that T is elementwise periodic on the cyclic elements of S on $S - L^*$. It must now be shown that each non-degenerate node has a finite period. If A is the least A-set which contains all the cut points then, by virtue of the fact that the cut points are an invariant set, A is invariant. Moreover A contains only degenerate nodes (end points). Thus, if E is a non-degenerate node and p is uniquely the cut point of S on E, then $E - p$ is a component K of $S - A$, and by component-wise periodicity, $E - p$ has a finite period. The sufficiency is now established.

For the purpose of proving the necessity, we assume that T is element-wise periodic on $S - L$, and that A is some T^n-invariant A-set, where n is fixed. Since, $A \subset \text{Int } C$, and the requirement that A be unbordered, are conditions on A that are not used immediately, we may, until the lemma is established, think of A as a true T^n-invariant A-set contained in another true A-set C. And, since n is fixed, the transformation $f = T^n$ may be used throughout most of the proof. In this connection it is to be understood that the terms, invariant, and orbit, now mean, f-invariant, and orbit under f. Obviously f is also elementwise periodic on $S - L$.

Suppose that Y is a set which contains every cyclic element E of S provided E has the property that the orbit of E lies in C. It follows that $E \subset Y$ implies that the orbit of E is also in Y; hence Y is invariant and contains A. Let B be the least A-set which contains Y.[15] Since Y is invariant and B is a minimal A-set, B is invariant. The minimal property for B also insures that $B \subset C$. Thus we have $A \subset B \subset C$ and $f(B) = B$.

We now establish a lemma. During the course of the proof it is convenient to use Theorem (2.6). This action is not improper since Theorem (3.2) is available as a means of showing that the necessary requirements are satisfied.

(3.41) LEMMA.   *If K is a component of $S - B$, there exists an integer n such that $f^n(K) \cdot C = 0$.*

Proof. Let $y = F(K)$ and assume that y is a bordered A-set in $\bar{K}$; in other words, suppose that there exists a true cyclic element E of $\bar{K}$ which contains the point y. Then, since E was not included in B, there exists an integer n such that $f^n(E) \cdot C = y$. And if $x \, \epsilon \, f^n(K) \cdot C$, the cyclic chain $C(x, y)$ is contained in the A-set C. This means there is a point $z \neq y$, such that $z \, \epsilon \, f^n(E) \cdot C$, which is impossible. Hence the point x does not exist, and $f^n(K) \cdot C = 0$ in this case.

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[15] The writer is indebted to G. T. Whyburn for suggesting that B be defined in this way. And, in general, for the privilege of seeing ATW in proof.

If, on the other hand, y is an unbordered A-set in $\bar{K}$, then y is an end point of $\bar{K}$. We now show that $f^n(K) \cdot C = 0$ when y is an end point. Suppose that K has period m and consider the homeomorphism $h = f^m$. Then $h(\bar{K}) = \bar{K}$ is elementwise periodic on $\bar{K} - L$ and leaves y fixed. By Theorem (2.6-w1) there exists a cut point $x \neq y$ in K such that $h(x) = x$; hence, each cyclic element in the chain $X = C(x, y)$ is fixed under h. Now let $\{y_i\}$ be a sequence of cut points in X such that $\operatorname{Lim} y_i = y$. Because $\{y_i\}$ consists of (infinitely many distinct) points of K, there exists an integer $n(i) \leqq m$ such that $f^{n(i)}(y_i)$ non-ϵC, for each i. Obviously, then, infinitely many of the $n(i)$ are equal. Hence we may choose some integer $n \leqq m$, and a subsequence $\{y_j\}$, such that $f^n(y_j)$ non-ϵC, for all j. If we suppose that there exists a $z \epsilon f^n(K) \cdot C$ and consider the cyclic chain $Y = C(z, f^n(y))$, we find that $Y \subset C$, because z and $f^n(y)$ are in C. On the other hand $f^n(y_j)$ can not be in the A-set C, because of the special choice of the sequence $\{y_j\}$. Thus no point z exists and the proof of the lemma is complete.

Returning to the proof of the necessity we observe that, by virtue of Lemma (3.41) and the inclusion $A \subset \operatorname{Int} C$, no point $y = F(K)$ is a point of A. It must therefore follow that $y \epsilon M$, where M is some component of $S - A$. Hence we have at our disposal a connection between the components K and M which reflects the statement that A is contained in the interior of C. This situation is exploited in two cases below; each of two possible ways in which $A \subset \operatorname{Int} B$ fails is shown to contradict the inclusion $A \subset \operatorname{Int} C$ at some point.

Suppose that there is a sequence $p_k \to p \epsilon A$, where $p_k \epsilon S - B$, for each k. Using the results of the preceding paragraph it is possible to select a subsequence $\{p_j\}$ such that $p_j \epsilon K(j)$, where $K(j)$ is a component of $S - B$; $K(j) \cdot K(i) = 0$, for $i \neq j$; and $p_j \to p$. Furthermore this sequence can be refined so that the cut points $z_j = F(K(j))$ form a sequence which also converges to p. This last statement rests on the property that the components $K(j)$ form a null sequence. Moreover it gives (indirectly) the convergence $\operatorname{Lim} K(j) = p$ which is needed below.

We now consider the case wherein an infinite number of points z_j are contained in some component M of $S - A$. Here a further refinement gives a sequence $\{K(i)\}$ such that $K(i) \subset M$, for all i. Thus, because *the hypothesis states that A is an unbordered A-set*, we find that $F(M) = x$ is an *end point* of $\bar{M}$ and that $\operatorname{Lim} \overline{K(i)} = x = p$. These properties will now be extended to larger sets of similar structure. Let $Q(z_i)$ be the sum, $\overline{K(i)} + f^m(\overline{K(i)}) + f^{2m}(\overline{K(i)}) + \cdots + f^{km}(\overline{K(i)})$, where $k = 0, 1, 2, \cdots, q/m$, with q the period for $K(i)$, and m the period for M. It is easily seen that

$Q(z_i)$ consists of exactly those sets in the orbit of $\overline{K(i)}$ which are also contained in M and contain $z_i = F(K(i))$. Moreover the period for $Q(z_i)$ is m, for all i. We may now state that $\operatorname{Lim} Q(z_i) = p$. The details of the short proof needed to establish this convergence can be supplied along the lines indicated (see null sequence) above. Now, using the images of the sequence of sets $Q(z_i)$, we get

$$(*) \qquad\qquad \operatorname{Lim} f^r(Q(z_i)) = f^r(p)$$

where r is fixed in the range 0 to m, $f^r(p)$ is an end point of $f^r(\bar{M})$, and convergence takes place in $f^r(\bar{M})$. In going this far we have used little of the weight of Lemma (3.41) from which we must derive our power. But, having prepared the way, we now select a convergent sequence $\{f^{s(n)}(K(n))\}$ such that $f^{s(n)}(K(n)) \cdot C = 0$ and $f^{s(n)}(\overline{K(n)}) \subset f^r(Q(z_n))$, for some fixed r. This statement is based largely on the fact that the range for i (which changes to n) is infinite while the range for r is finite. It follows, using $(*)$, that $\operatorname{Lim} f^{s(n)}(K(n)) = \operatorname{Lim} f^r(Q(z_n)) = f^r(p) \,\epsilon\, A$. This is impossible in view of the fact that $f^r(p)$ is interior to C.

It remains to be shown that the second case, wherein each $K(i)$ is uniquely contained in some component $M(i)$ of $S - A$, is also impossible. Here we again use Lemma (3.41) to get a convergent subsequence with the property $f^{s(n)}(K(n)) \cdot C = 0$. The corresponding sets $f^{s(n)}(M(n))$ form a null sequence. And we see at once that $\operatorname{Lim} f^{s(n)}(\overline{M(n)}) = x$ implies $x = p$, since $p_n \to p$ and $z_n \to p$ hold. But p is interior to C, and $x = p$ is impossible in view of the special choice of the sets $f^{s(n)}(K(n))$. Thus $A \subset \operatorname{Int} B$ can not fail to hold in either case.

Since the componentwise perodicity of T is obvious, this completes the proof of the theorem.

It seems worth while to note that when A, B, and C exist, and B satisfies Lemma (3.41), then the fact that $A \subset \operatorname{Int} C$ implies $A \subset \operatorname{Int} B$ is a consequence of this method of proof.

3.42) COROLLARY. *If T is pointwise periodic* [16] *on $S - L$ where S is a dendrite and p is a cut point of period n, then there exists a sequence of regions (connected and open sets) R_i closing down on p and such that, for each i, $\bar{R}_i$ is an A-set invariant under T^n.*

4. Summary of characterizations. Although the eight characterizations that follow are concerned exclusively with elementwise periodicity on $S - L$,

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[16] Each point has a finite orbit.

many of these may be altered slightly to make them applicable to either S or $S - L^*$.

(4.1) DEFINITION. Let $I(n)$ denote the sum of all cyclic elements in S which are invariant under $T^n(S) = S$. If there exists a sequence of integers $n_1, n_2, \cdots$ such that: (a) $I(n_1) \subset I(n_2) \subset \cdots$; (b) $\Sigma I(n_i) \supset S - L$; (c) for any $\epsilon > 0$ there exists an i such that each component of $S - I(n_i)$ is of diameter less than ϵ; then we say that T admits an expanding approximation to $S - L$.

(4.2) THEOREM. In order that the homeomorphism $T(S) = S$ be elementwise periodic on the cyclic elements of S in $S - L$ it is necessary and sufficient that one of the following conditions be satisfied:

(a) T admits an expanding approximation to $S - L$.

(b) T is componentwise periodic and each unbordered T^n-invariant A-set A in S admits a contracting T^n-approximation which preserves interiority at A.

(c-g) T is componentwise periodic and, for every positive integer n, T^n has one of the five properties [17] of Ayres and Whyburn.

(h) If M is any set in $S - L$ having the property that M contains each cyclic element of S which intersects [18] M then $T(M) \subset M$ implies $T(M) = M$.

Remarks. No formal proofs are needed here: see Theorem [19] [4.7w]; Theorem (3.4); Theorems (3.2) and (2.6); and Theorem [1.2w]. In using these, change L^* to L, property (α) to componentwise periodicity, and point to cyclic element when the situation warrants such a change.

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[17] Four defined in (2.5); the fifth is the fixed point property for nodal sets. They are defined for T and T-invariance; if T^n is used, T^n-invariance must be used in order to apply Theorem (2.6). It is not overly strong to assume these properties for *every* positive integer n; the reader may readily verify that neither of the assumptions, (a2), for $n = 1$ and $n = 2$, implies the other.

[18] M is a sum of cyclic elements.

[19] ATW p. 249 and the proof on the next page. The necessity argument requires [4.3W] which we shall have independently in (c-g) below. The sufficiency is immediate. No cycle of proofs was attempted.

5. Continuous decomposition; concluding remarks. If each point of S has a finite orbit then $T(S) = S$ is said to be pointwise periodic, and the use of the orbit space, wherein each orbit is considered as a point, is often advantageous. For this purpose it is desirable to know that the limit of a convergent sequence of orbits is an orbit; the associated transformation is then interior (open) and the decomposition is said to be continuous. It was originally intended that this section should contain a proof of the following theorem: [20] *If the period function of the homeomorphism $T(S) = S$ is defined for points and is bounded on each cyclic element in S then the orbit decomposition for S is continuous.* Two interesting theorems [21] recently published combine to include this result but they do not answer certain conjectures which seek to make the continuous decomposition property cyclically extensible.[22] The large amount of necessary ground work makes a competent treatment of such conjectures lie beyond the scope of this section; instead we use a special continuum to illustrate briefly the use of Theorem (3.4) in this direction. The invariance of the non-degenerate element below makes for brevity as much as do other more apparent assumptions.

We let X denote a semi-locally-connected continuum containing only one true cyclic element E. It is also assumed that the homeomorphism $T(X) = X$ is elementwise periodic on X with $T(E) = E$, and that $[G]$ is a collection of disjoint closed invariant sets filling up X subject to the conditions: (1) If $x \in X - E$, or $x \in E$ is a cut point of X, then $x \in G_x \in [G]$ implies that G_x is the orbit of x; (2) $y \in E$ implies that the sets $[G_y]$ which (as a consequence of (1)) fill up E give a continuous decomposition of E. It is asserted that $[G]$ *is a continuous decomposition of X.* Using distinct sets of the decomposition suppose that $p_i \in G(i) \in [G]$, $p_i \to p$, $G(i) \to L$ and $p \in L$. We wish to show that $p \in G_p \in [G]$ implies $L = G_p$. If p has period n and lies in $X - E$ then the sets $G(i)$ are point-orbits. Moreover p is an unbordered T^n-invariant A-set; hence if $M(i)$ denotes the orbit of p_i under T^n we may show by Theorem (3.4) that $M(i) \to p$. Thus $G(i)$ is the sum of at most n sets $T^k(M(i))$ and $T^k(M(i)) \to T^k(p)$. The result $G_p = L$ now follows by standard methods. Turning to the other possibility, $p \in E$, we see that the fact that $p_i \in E$ for infinitely many i, gives the desired result by the continuous decomposition

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[20] Abstract, *Bulletin of the American Mathematical Society*, vol. 46 (1939), p. 82.
[21] ATW [5.1] p. 251 and [6.42] p. 258. The first due to Whyburn, the second due to Hall and Kelley. In this connection see Montgomery § 7, p. 262, and [7.11] for n-dimensional manifolds.
[22] This means that the property holds for all of S provided it holds for the *orbit* of each cyclic element in S. These conjectures are concerned with decompositions into sets related to orbits; they may be, for example, the closures of orbits.

on E. If on the other hand K is a component of $X - E$ and $p_i \, \epsilon \, K$ holds for infinitely many i, then Theorem (3.4) may be applied to the set consisting of p and the orbit of K under $T^n(p) = p$. Here we use $M(i) \to p$ as done above. Finally with $p_i \, \epsilon \, K(i)$ in one-to-one correspondence and each $K(i)$ a component of $X - E$ we may show that $L = G_p$ by structural considerations. Since the sets $K(i)$ form a null sequence it can be shown that $\operatorname{Lim} G(i) = \operatorname{Lim} O(i)$ where $O(i)$ is the orbit of $z_i = F'(K(i))$. (This remark ignores certain necessary refinements). Thus the continuity of the decomposition on E may be used in view of the inclusion $O(i) \subset E$.

In conclusion it may be noted that since little is said about the orbits of end points of S it would be fitting to isolate the structural features of S that accompany the infinite orbit in L. However the work in section (2) suggests a systematic analysis of the onto-homeomorphism and there is evidence that this question will automatically fall within the scope of any investigation of that nature.

UNIVERSITY OF MISSOURI,
        AND
UNIVERSITY OF VIRGINIA (Summer Quarter).

ON NON- NEGATIVE FUNCTIONAL TRANSFORMATIONS.*

By Erich Rothe.

1. Introduction. Alexandroff and Hopf [1] have given a topological proof of a theorem—known as the theorem of Frobenius—whose essential part may be stated as follows: let E be an n-dimensional real space and let $\mathfrak{y} = \mathfrak{F}(\mathfrak{x})$ be a homogeneous linear transformation mapping the point $\mathfrak{x} \subset E$ with coördinates $x_1, x_2, \cdots, x_n$ into the point $\mathfrak{y} \subset E$ with coördinates $y_1, y_2, \cdots, y_n$; if (i) the coefficients of the matrix of $\mathfrak{F}$ are non-negative and if (ii) the determinant of $\mathfrak{F}$ is different from zero, then there exists at least one positive eigen-value of $\mathfrak{F}$, i. e. a positive number λ such that $\mathfrak{x} = \lambda \mathfrak{F}(\mathfrak{x})$ for at least one point $\mathfrak{x}$ whose coördinates are not all zero.

Let us call a point $\mathfrak{x} \subset E$ non-negative if its coördinates are non-negative, and a transformation $\mathfrak{F}$ non-negative if the image of any non-negative point $\mathfrak{x}$ is non-negative. With these terms the above theorem may be restated in the following form: the linear transformation $\mathfrak{F}(\mathfrak{x})$ has at least one positive eigen-value if (i) $\mathfrak{F}$ is non-negative and if (ii) there exists a positive number m such that for all $\mathfrak{x}$

$$\| \mathfrak{y} \| = \| \mathfrak{F}(\mathfrak{x}) \| \geqq m \| \mathfrak{x} \|$$

where

$$\| \mathfrak{x} \| = + \sqrt{x_1{}^2 + x_2{}^2 \cdots + x_n{}^2}, \qquad \| \mathfrak{y} \| = + \sqrt{y_1{}^2 + y_2{}^2 \cdots + y_n{}^2}.$$

It is the object of the present paper to prove similar theorems for certain non-linear transformations in Hilbert spaces whose points $\mathfrak{x}$ are functions $\mathfrak{x} = x(t)$. In such spaces we might call the point $\mathfrak{x} = x(t)$ non-negative if $x(t) \geqq 0$ for all t for which $x(t)$ is defined, or we might call $x(t)$ non-negative if all components of $x(t)$ with respect to a given complete system of orthogonal functions are non-negative. With either definition we shall prove a theorem concerning the existence of a positive eigen-value for certain non-linear completely continuous [2] transformations $\mathfrak{F}(\mathfrak{x})$ mapping non-negative points into non-negative points (Theorems 4. 1 and 4. 2). These theorems are applications of a general eigen-value theorem in the abstract Hilbert space E

[1] [1], p. 480. (Numbers in brackets refer to the bibliography).

[2] I. e. a continuous transformation mapping each bounded set into a compact set.

(Theorem 3. 2).[3]   This general theorem is based on a fixed-point theorem for
the mapping on itself of a " convex " set situated on a sphere of E (Theorem
3. 1; for the definition of convexity on a sphere of E cf. **2** IV).   This fixed-
point theorem is obtained in turn by introducing a " stereographic projection "
(**2** III) and thus mapping s on a convex set $\bar{s}$ lying in a " plane " space.   For
$\bar{s}$ we can then apply a well known fixed-point theorem of Schauder.[4]

As an application of the abstract theory two eigen-value theorems con-
cerning *non-linear* integral equations, are stated in **5**.   (As to the relation to
a result of Birkhoff and Kellogg, [2], see the second paragraph of **5**).   The
case of a *linear* integral equation, however, escapes our analysis.   It would be
of interest to have a topological proof for the theorem of Jentzsch [5] which
states the existence of a positive eigen-value for a linear integral equation
with a continuous positive kernel and is thus a direct generalization of the
theorem of Frobenius mentioned above.   It would also be desirable to have
such a proof in the classical case of a symmetric kernel.

2.   Preliminaries.   Let E be the real Hilbert space.   For future reference
we recall the following properties of the scalar product $(\mathfrak{x}, \mathfrak{y})$ of two points
$\mathfrak{x}, \mathfrak{y}$ of E:

$$(\mathfrak{x}, \mathfrak{y}) = (\mathfrak{y}, \mathfrak{x})$$
$$(\mathfrak{x}, \mathfrak{x}) > 0 \text{ if } \mathfrak{x} \text{ is not the zero element of } E$$
(2.1) $$(\mathfrak{x}, \mathfrak{x}) = 0 \text{ if } \mathfrak{x} \text{ is the zero element of } E$$
$$(\mathfrak{x} + \mathfrak{y}, \mathfrak{z}) = (\mathfrak{x}, \mathfrak{z}) + (\mathfrak{y}, \mathfrak{z})$$
$$(\tau\mathfrak{x}, \mathfrak{y}) = \tau(\mathfrak{x}, \mathfrak{y}) \text{ for any real number } \tau.[6]$$

If the norm $\| \mathfrak{x} \|$ of $\mathfrak{x}$ is defined as $+ \sqrt{(\mathfrak{x}, \mathfrak{x})}$, then Schwarz' inequality
$|(\mathfrak{x}, \mathfrak{y})| \leqq \| \mathfrak{x} \| \cdot \| \mathfrak{y} \|$ holds, the equality being true only if $\alpha\mathfrak{x} + \beta\mathfrak{y} = 0$ for
some α, β with $\alpha^2 + \beta^2 \neq 0$.   Moreover the scalar product is continuous.

The main object of this section is, in close analogy to the n-dimensional
case, to introduce a stereographic projection in E and to prove some of its
properties which will be useful later.

I.   If $\mathfrak{p}$ and $\mathfrak{x}^0$ are two points of E, the set

(2. 2) $$\mathfrak{x} = \mathfrak{p} + \lambda(\mathfrak{x}^0 - \mathfrak{p})$$ $(- \infty < \lambda < \infty)$

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[3] Regarding the relation of this theorem to eigen-value theorems first proved by
Birkhoff and Kellogg [2] and stated later with greater generality by Schauder, [5],
p. 179, and Rothe [4], §§ 3 and 4, see footnote 10.

[4] [5], p. 174.   The first proof for a fixed-point theorem in function-spaces is con-
tained in [2].

[5] [3], p. 235.

[6] German letters denote points of E, Greek letters real numbers.

is called the straight line determined by $\mathfrak{x}^0$ and $\mathfrak{p}$; the subset for which $0 \leq \lambda \leq 1$ is called the *segment* $\overline{\mathfrak{p}\mathfrak{x}^0}$. If E' is a subset of E which has one and only one intersection $\tilde{\mathfrak{x}}$ different from $\mathfrak{p}$ with the straight line (2.2), $\tilde{\mathfrak{x}}$ is called the *projection of $\mathfrak{x}^0$ from $\mathfrak{p}$ on E'*.

II. Let r be a positive number and $\mathfrak{b}$ a point of E. By definition, the *sphere* with radius r and center $\mathfrak{b}$ is the set of all points $\mathfrak{x}$ for which $\| \mathfrak{x} - \mathfrak{b} \| = r$. The set of all points $\mathfrak{x}$ for which $\| \mathfrak{x} - \mathfrak{b} \| \leq r$ is called the *full sphere* with radius r and center $\mathfrak{b}$. A sphere will be denoted by S and the corresponding full sphere by V. The segment determined by two points $\mathfrak{x}^0$ and $\mathfrak{x}^1$ of S (cf. I) is also referred to as the *chord* $\overline{\mathfrak{x}^0\mathfrak{x}^1}$. It is readily seen that all points of the chord $\overline{\mathfrak{x}^0\mathfrak{x}^1}$, with the exception of $\mathfrak{x}^0$ and $\mathfrak{x}^1$, are interior points of V. Let S be the sphere with radius r whose center is the zero element $\mathfrak{o}$ of E, let $\mathfrak{p}$ be a point of S, and $\mathfrak{x}^1$ an arbitrary point of E different from $\mathfrak{p}$ for which, moreover, $(\mathfrak{p}, \mathfrak{p} - \mathfrak{x}_0) \neq 0$. By the use of the rules (2.1) it is easily seen that the projection $\tilde{\mathfrak{x}}^0$ of $\mathfrak{x}^0$ from $\mathfrak{p}$ on S (cf. I) is given by (2.2) with

$$(2.3) \qquad \lambda = \frac{2(\mathfrak{p}, \mathfrak{p} - \mathfrak{x}^0)}{(\mathfrak{x}^0 - \mathfrak{p}, \mathfrak{x}^0 - \mathfrak{p})} = \frac{r^2 - (\mathfrak{x}^0, \mathfrak{p})}{(\| \mathfrak{x}^0 \|^2 + r^2)/2 - (\mathfrak{x}^0, \mathfrak{p})} \;.$$

III. Let S be the sphere $\| \mathfrak{x} \| = r$ and $\mathfrak{a}$ a point of S. By definition, the *tangent plane* to S at $\mathfrak{a}$ is the set of all points $\mathfrak{x} = \mathfrak{a} + \lambda \mathfrak{v}$ where λ is a real variable $(-\infty < \lambda < \infty)$, and $\mathfrak{v}$ a point subject to the conditions

$$(2.4) \qquad \qquad \| \mathfrak{v} \| = 1, \qquad (\mathfrak{v}, \mathfrak{a}) = 0.$$

For any point $\mathfrak{x}$ of S different from $\mathfrak{p} = -\mathfrak{a}$, the *stereographic projection* $\bar{\mathfrak{x}}$ of $\mathfrak{x}$ from $\mathfrak{p}$ is then defined as the projection (cf. I) of $\mathfrak{x}$ from $\mathfrak{p}$ on the tangent plane T at $\mathfrak{a}$; $\mathfrak{p}$ is called the pole of the stereographic projection. For later use we note the following formulas for the stereographic projection which follow easily from the definition and the rules (2.1):[7]

$$(2.5) \qquad \bar{\mathfrak{x}} = \mathfrak{p} + \bar{\mu}(\mathfrak{x} - \mathfrak{p}) = \mathfrak{a} + \bar{\lambda}\mathfrak{v}$$

where

$$(2.6) \qquad \bar{\mu} = 4r^2/\| \mathfrak{x} - \mathfrak{p} \|^2, \qquad \bar{\lambda} = \bar{\mu}(\mathfrak{v}, \mathfrak{x} - \mathfrak{p}) = \bar{\mu}(\mathfrak{v}, \mathfrak{x})$$

and

$$(2.7) \qquad \qquad \mathfrak{x} = \mathfrak{p} + \mu(\bar{\mathfrak{x}} - \mathfrak{p})$$

where $\mu = 4r^2/\| \mathfrak{p} - \bar{\mathfrak{x}} \|^2$.

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[7] Noticing that $(\mathfrak{p}\bar{\mathfrak{x}}) = -\mathfrak{a}(\mathfrak{a} + \lambda\mathfrak{v}) = -\mathfrak{a}^2 = -r^2$, and that $2\mathfrak{p}(\mathfrak{p} - \mathfrak{x}) = \mathfrak{p}^2 + \bar{\mathfrak{x}}^2 - 2\mathfrak{p}\mathfrak{x} = (\mathfrak{x} - \mathfrak{p})^2$, one obtains the value of $\bar{\mu}$ in (2.6) upon multiplying (2.5) by $\mathfrak{p}$; the value of λ follows then by multiplying (2.5) by $\mathfrak{v}$. Finally, the value of μ is obtained by multiplying (2.7) by $\bar{\mathfrak{x}} - \mathfrak{p}$ and noticing that (2.5) and (2.6) yield $(\bar{\mathfrak{x}} - \mathfrak{p}, \mathfrak{x} - \mathfrak{p}) = \bar{\mu} \| \mathfrak{x} - \mathfrak{p} \|^2 = 4r^2$.

IV. A subset s of the sphere S not containing the point $\mathfrak{p} \subset S$ is called *convex with respect to* $\mathfrak{p}$ *if* $\mathfrak{x}^0 \subset s$, $\mathfrak{x}^1 \subset s$ implies that the projection of the chord $\overline{\mathfrak{x}^0\mathfrak{x}^1}$ from $\mathfrak{p}$ on S belongs to s.[8] A set $\bar{s}$ in the tangent plane T (cf. III) to S is called *convex* if $\bar{\mathfrak{x}}^0 \subset \bar{s}$, $\bar{\mathfrak{x}}^1 \subset \bar{s}$ implies that the segment $\overline{\mathfrak{x}^0\mathfrak{x}^1}$ belongs to $\bar{s}$. If $\bar{\mathfrak{x}}^0$ and $\bar{\mathfrak{x}}^1$ are the stereographic projections from $\mathfrak{p}$ of $\mathfrak{x}^0$ and $\mathfrak{x}^1$, and if arc $\mathfrak{x}^0\mathfrak{x}^1$ denotes the projection of the chord $\overline{\mathfrak{x}^0\mathfrak{x}^1}$ from $\mathfrak{p}$ on S, then the segment $\overline{\bar{\mathfrak{x}}^0\bar{\mathfrak{x}}^1}$ is the stereographic projection of arc $\mathfrak{x}^0\mathfrak{x}^1$ from $\mathfrak{p}$. Consequently: *the stereographic projection $\bar{s}$ of a set $s \subset S$ which is convex with respect to* $\mathfrak{p}$ *is convex.*

V. Let $\mathfrak{x}^0$, $\mathfrak{x}^1$ be two points of the sphere S and let $\bar{\mathfrak{x}}^0$, $\bar{\mathfrak{x}}^1$ be their stereographic projections from the point $\mathfrak{p} \subset S$:

$$(2.9) \qquad \bar{\mathfrak{x}}^0 = \mathfrak{p} + \bar{\mu}^0(\mathfrak{x}^0 - \mathfrak{p}), \qquad \bar{\mathfrak{x}}^1 = \mathfrak{p} + \bar{\mu}^1(\mathfrak{x}^1 - \mathfrak{p})$$

where $\bar{\mu}^0$, $\bar{\mu}^1$ are the corresponding $\bar{\mu}$-values given by (2.6). Then the inequalities

$$(2.10) \qquad \qquad \| \mathfrak{x}^i - \mathfrak{p} \| \geqq d > 0 \qquad \qquad (i = 0, 1)$$

imply

$$(2.11) \qquad \qquad \| \mathfrak{x}^0 - \bar{\mathfrak{x}}^1 \| \leqq (32 r^4/d^4) \, \| \mathfrak{x}^0 - \mathfrak{x}^1 \|.$$

Proof. It follows from (2.9) that

$$(2.12) \qquad \| \bar{\mathfrak{x}}^0 - \bar{\mathfrak{x}}^1 \| \leqq | \bar{\mu}^0 | \, \| \mathfrak{x}^0 - \mathfrak{x}^1 \| + | \bar{\mu}^0 - \bar{\mu}^1 | \, \| \mathfrak{x}^1 - \mathfrak{p} \| ;$$

on the other hand, it follows from (2.6), (2.10), and Schwarz' inequality that

$$| \bar{\mu}^0 - \bar{\mu}^1 | = \frac{8 r^2 \, | (\mathfrak{x}^0 - \mathfrak{x}^1, \mathfrak{p}) |}{\| \mathfrak{x}^0 - \mathfrak{p} \|^2 \, \| \mathfrak{x}' - \mathfrak{p} \|^2} \leqq \frac{8 r^3}{d^4} \, \| \mathfrak{x}^0 - \mathfrak{x}^1 \|.$$

Therefore, from (2.12)

$$\| \bar{\mathfrak{x}}^0 - \bar{\mathfrak{x}}^1 \| \leqq \| \mathfrak{x}^0 - \mathfrak{x}^1 \| \, \{ \bar{\mu}^0 + (8 r^3/d^4) \, \| \mathfrak{x}^1 - \mathfrak{p} \| \}.$$

This proves (2.11) since $\| \mathfrak{x}^1 - \mathfrak{p} \| \leqq 2 r$ and, on account of (2.6) and (2.10), $| \bar{\mu}^0 | \leqq (4 r^2/d^2) \leqq (16 r^4/d^4)$.

VI. Let s be a subset of the sphere S which has a positive distance from the point $\mathfrak{p} \subset S$. Then the stereographic projection $\bar{s}$ of s from $\mathfrak{p}$ is a bounded set. This follows immediately from (2.5) and (2.6).

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[8] A set which is convex with respect to $\mathfrak{p}$ is not necessarily convex with respect to a point $\mathfrak{p}'$ different from $\mathfrak{p}$. For instance the projection from $\mathfrak{p}$ on S of the chord $\mathfrak{x}^0\mathfrak{x}^1$ is convex with respect to $\mathfrak{p}$ but certainly not with respect to all points of the sphere S.

VII. The following two facts are immediate consequences of V and VI:

(i) let $\mathfrak{x}^1, \mathfrak{x}^2, \cdots$ be a sequence of points on S which is convergent and has a positive distance from the point $\mathfrak{p} \subset S$. Then the sequence $\overline{\mathfrak{x}^1}, \overline{\mathfrak{x}^2}, \cdots$ of the stereographic projections from $\mathfrak{p}$ is also convergent.

(ii) Let s be a subset of the sphere S not containing the point $\mathfrak{p}$ of S, and $\mathfrak{F}(\mathfrak{x})$ a completely continuous transformation defined in s with range in S and such that $\| \mathfrak{F}(\mathfrak{x}) - p \| \geqq d > 0$ for all $\mathfrak{x} \subset s$. Denote, for any $\mathfrak{x} \subset S - \mathfrak{p}$, the stereographic projection of $\mathfrak{x}$ from $\mathfrak{p}$ by $r = \mathfrak{P}(\mathfrak{x})$, and its inverse by $\mathfrak{P}^{-1}(\mathfrak{x})$. Then, the transformation $\mathfrak{P}\mathfrak{F}\mathfrak{P}^{-1}(\mathfrak{x})$ is completely continuous.

VIII. The tangent plane T (cf. III) to the sphere S at the point $\mathfrak{a}$ becomes a linear space if, in an obvious manner, we define the T-sum of two points $\mathfrak{x}$ and $\mathfrak{y}$ of T by

$$[\mathfrak{x} + \mathfrak{y}]_T = \mathfrak{a} + (\mathfrak{x} - \mathfrak{a}) + (\mathfrak{y} - \mathfrak{a})$$

and the T-product of $\mathfrak{x}$ with a real number α by

$$[\alpha \mathfrak{x}]_T = \mathfrak{a} + \alpha(\mathfrak{x} - \mathfrak{a}).$$

If, moreover, the T-scalar product is defined by

$$(\mathfrak{x}, \mathfrak{y})_T = (\mathfrak{x} - \mathfrak{a}, \mathfrak{y} - \mathfrak{a})$$

and the T-norm by

$$\| \mathfrak{x} \|_T = + \sqrt{(\mathfrak{x} - \mathfrak{a}, \mathfrak{x} - \mathfrak{a})},$$

T becomes a Hilbert space.

The notions of a straight line or segment determined by two points $\mathfrak{x}, \mathfrak{y}$ of T, of the boundedness of a set of T, of the continuity of a transformation mapping a set of T into a set of T, or of the complete continuity of such a transformation may all be defined either in terms of the space E or in terms of the space T. It is, however, readily seen that these two definitions actually coincide.

3. Existence theorems in the Hilbert space.

THEOREM 3. 1. *Let S be the sphere $\| \mathfrak{x} \| = r$, $\mathfrak{p}$ a point of S, and s a closed subset of S which is convex with respect to $\mathfrak{p}$ (cf. **2** IV). Let $\mathfrak{y} = \mathfrak{F}_1(\mathfrak{x})$ be a transformation defined for $\mathfrak{x} \subset s$ with the following properties*: (i) $\mathfrak{F}_1$ *is completely continuous*; (ii) *the image of s is contained in s. Then there exists at least one fixed-point of $\mathfrak{F}_1$, i. e. a point $\mathfrak{x} \subset s$ for which $\mathfrak{x} = \mathfrak{F}_1(\mathfrak{x})$.*

Proof. Let $\overline{\mathfrak{x}} = \mathfrak{P}(\mathfrak{x})$ be the stereographic projection of the point

$\mathfrak{x} \subset S - \mathfrak{p}$ from $\mathfrak{p}$ on the tangent plane T at $-\mathfrak{p}$. Denote the stereographic projection of s by $\bar{s}$, and the transformation $\mathfrak{P}\mathfrak{F}_1\mathfrak{P}^{-1}(\mathfrak{x})$, which is defined in $\bar{s}$, by $\mathfrak{G}(\bar{\mathfrak{x}})$. Since T is a Hilbert space (cf. **2** VIII) and, therefore, normed, linear and complete, we see immediately from **2** IV, VI, VII, VIII, and the hypotheses of our theorem that $\bar{s}$ is a convex, closed, and bounded subset of T and that $\mathfrak{G}(\bar{\mathfrak{x}})$ is a completely continuous transformation which maps $\bar{s}$ in itself. Hence, according to a theorem of J. Schauder,[9] this transformation possesses a fixed point $\bar{\mathfrak{x}}$. Obviously $\mathfrak{x} = \mathfrak{P}^{-1}(\bar{\mathfrak{x}})$ is then a fixed point of $\mathfrak{F}_1$.

THEOREM 3.2. *Let* $\mathfrak{y} = \mathfrak{F}(\mathfrak{x})$ *be a transformation defined for all* $\mathfrak{x}$ *of a certain subset* E_1 *of* E. *We assume that there exists a certain sphere* S *defined by* $\| \mathfrak{x} \| = r$, *a point* $\mathfrak{p}$ *on* S, *and a closed subset* s *of* S *which belongs to* E_1 *and is convex with respect to* $\mathfrak{p}$ *such that for* $\mathfrak{x} \subset s$ *the following conditions hold:* (i) $\mathfrak{F}(\mathfrak{x})$ *is completely continuous and there exists a positive constant* m *such that* $\| \mathfrak{F}(\mathfrak{x}) \| \geqq m$. (ii) $r\mathfrak{F}(\mathfrak{x})/\| \mathfrak{F}(\mathfrak{x}) \|$ *belongs to* s.

Then there exists a positive number λ *and a point* $\mathfrak{x}$ *of* s *such that*

(3.1) $\mathfrak{x} = \lambda\mathfrak{F}(\mathfrak{x}).$[10]

Proof. For $\mathfrak{x} \subset s$ the mapping $\mathfrak{F}_1(\mathfrak{x}) = r\mathfrak{F}(\mathfrak{x})/\| \mathfrak{F}(\mathfrak{x}) \|$ satisfies all the hypotheses of Theorem 3.1. Therefore $\mathfrak{x} = \mathfrak{F}_1(\mathfrak{x})$ for a certain $\mathfrak{x} \subset s$. For this $\mathfrak{x}$ equation (3.1) holds with $\lambda = r/\| \mathfrak{F}(\mathfrak{x}) \|$.

4. Application to Analysis. In what follows E will be the space L^2 of all functions $\mathfrak{x} = x(t)$ which together with their squares are integrable in $0 \leqq t \leqq 1$ in the sense of Lebesgue. As usual the scalar product of two such functions $\mathfrak{x} = x(t)$, $\mathfrak{y} = y(t)$ is defined by

$$(\mathfrak{x}, \mathfrak{y}) = \int_0^1 x(t)y(t)dt \quad \text{and} \quad \| \mathfrak{x} \| = \sqrt{(\mathfrak{x}, \mathfrak{x})}.$$

As a preparation for the next theorem we prove the following

LEMMA 4.1. *Let* S *be the sphere* $\| \mathfrak{x} \| = r$ *in* $E = L^2$, $\mathfrak{p}$ *that point of* S *which represents the constant* $-r$, *and* s *the set of all points* $\mathfrak{x} = x(t)$ *of* S *for which* $x(t) \geqq 0$ *almost everywhere in* $0 \leqq t \leqq 1$. *Then* (i) s *is convex with respect to* $\mathfrak{p}$ (cf. **2** IV) *and* (ii) s *is closed.*

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[9] [5], p. 174.
[10] The theorems mentioned in footnote 3 deal with the case in which the set s is identical with the whole sphere S so that condition (ii) of our theorem is automatically satisfied while the inequality $\| \mathfrak{F}(\mathfrak{x}) \| \geqq m$ holds for *all* $\mathfrak{x} \subset S$. This latter fact restricts the applicability of these theorems as compared with Theorem 3.2. Cf. the remarks in the second paragraph of section **5** of the present paper.

Proof. Let $\mathfrak{x}^0 = x^0(t)$ and $\mathfrak{x}^1 = x^1(t)$ be two different points of s, and

$$(4.1) \qquad x^2(t) = \mathfrak{x}^2 = \alpha^0 \mathfrak{x} + \alpha^1 \mathfrak{x}^1 \qquad (\alpha^0 + \alpha^1 = 1; \, 0 < \alpha^0, \, 0 < \alpha^1)$$

a point of the chord $\overline{x\, x^1}$ different from x^0 and x^1. If then

$$(4.2) \qquad \bar{x}^2(t) = \bar{\mathfrak{x}}^2 = \mathfrak{p} + \lambda(\mathfrak{x}^2 - \mathfrak{p}) = \lambda x^2(t) + r(\lambda - 1)$$

is the projection of $\mathfrak{x}^2$ from $\mathfrak{p}$ on S (cf. **2** I), we have to prove that $\mathfrak{x}^2 \subset s$, i. e. that

$$(4.3) \qquad \bar{\mathfrak{x}}^2(t) \geqq 0 \qquad\qquad \text{(almost everywhere in } 0 \leqq t \leqq 1).$$

To see this we note that $\mathfrak{x}^2$ is an interior point of the full sphere $\| \mathfrak{x} \| \leqq r$, i. e. that $\| \mathfrak{x}^2 \| < r$. The formula (2.3) for λ in **2** II shows then that $\lambda > 1$. Hence we have from (4.2), (4.1), and the hypotheses concerning $x^0(t)$ and $x^1(t)$

$$\bar{x}^2(t) = \lambda\{\alpha^0 x^0(t) + \alpha^1 x^1(t)\} + r(\lambda - 1) \geqq r(\lambda - 1) \geqq 0$$

which proves (4.3) and therefore the assertion (i) of the lemma.

To prove the assertion (ii) we have to show the following: if $\mathfrak{x}^n = x^n(t)$ $(n = 1, 2 \cdots)$ considered as a sequence of points in $E \equiv L^2$ is convergent, if, moreover, $x^n(t) \geqq 0$ almost everywhere in $0 \leqq t \leqq 1$, and

$$(4.4) \qquad x(t) = \mathfrak{x} = \lim_{n \to \infty} \mathfrak{x}^n, \text{ i. e. } \lim_{n \to \infty} \int_0^1 [x(t) - x^n(t)]^2 dt = 0,$$

then $x(t) \geqq 0$ almost everywhere in $0 \leqq t \leqq 1$. Obviously it will be sufficient to prove that for each positive number ϵ the measure $m(e)$ of the set e of points t $(0 \leqq t \leqq 1)$ for which $x(t) \leqq -\epsilon$ is zero. But the fact that $m(e)$ = zero follows immediately from (4.4) and the inequality $x^n(t) \geqq 0$ since

$$\int_0^1 [x(t) - x^n(t)]^2 dt \geqq \int_e [x(t) - x^n(t)]^2 dt \geqq \epsilon^2 m(e)$$

for $n = 1, 2, \cdots$.

THEOREM 4. 1. *Let $\mathfrak{y} = \mathfrak{F}(x)$ be a completely continuous transformation mapping each point $\mathfrak{x} = x(t)$ of a certain subset E_1 of the space $E = L^2$ into a point $\mathfrak{y} = y(t) \subset L^2$. We assume that there exist two positive numbers r and m such that all functions $\mathfrak{x} = x(t) \subset L^2$ for which*

$$(4.5) \qquad\qquad \int_0^1 x^2(t) dt = r^2 \text{ and } x(t) \geqq 0$$

nearly everywhere in $0 \leqq t \leqq 1$ belong to E_1. Moreover, (4.5) implies: (i) $\| \mathfrak{F}(\mathfrak{x}) \| \geqq m$; (ii) $y(t) = F(x(t)) \geqq 0$.

Under these assumptions there exists a function $x(t)$ satisfying (4.5)
and a positive number λ such that $x(t) = \lambda \mathfrak{F}(x(t))$.

Proof. Let S be the sphere $\| \mathfrak{x} \| = r$ of $E = L^2$, and s that subset of S
for which also the second condition (4.5) holds. Let $\mathfrak{p}$ be the point $-r$ of S.
On account of Lemma 4.1 it is immediately seen that the assumptions of
our theorem imply those of Theorem 3.2. Hence Theorem 4.1 is a con-
sequence of Theorem 3.2.

In the same way the following Theorem 4.2 can be shown to be a con-
sequence of Theorem 3.2. Though the proofs of these two theorems are thus
essentially the same it should be noted that their analytical contents are dif-
ferent inasmuch as they deal with quite different types of functions.

THEOREM 4.2. *For* $\mathfrak{x} = x(t) \subset L^2$ *let*

$$x_n = \int_0^1 x(t) \phi_n(t) dt \qquad\qquad (n = 0, 1, 2, \cdots)$$

*be the components (Fourier coefficients) with respect to the normed orthogonal
and complete system*

(4.6) $\phi_0(t), \phi_1(t), \cdots$

*of functions in L^2. Let $\mathfrak{y} = \mathfrak{F}(\mathfrak{x})$ be a completely continuous transformation
mapping each point $\mathfrak{x} = x(t)$ of a certain subset E_1 of the space $E = L^2$ into
a point $\mathfrak{y} = y(t) \subset L^2$. We assume that there exist two positive numbers
r and m such that all functions $x(t) \subset L^2$ for which*

(4.7) $\displaystyle\sum_{n=0}^{\infty} x_n{}^2 = r^2$ *and* $x_n \geqq 0$ \qquad\qquad $(n = 0, 1, \cdots)$

belong to E_1. Moreover (4.15) *implies:* (i) $\| \mathfrak{F}(\mathfrak{x}) \| \geqq m$; (ii) $y_n \geqq 0$
$(n = 0, 1, \cdots)$ *where y_n are the components of $y(t) = \mathfrak{F}(x(t))$ with respect
to the system* (4.6). *Under these assumptions there exists a function $x(t)$
whose components x_n satisfy* (4.7) *and a positive number λ such that $x(t) =
\lambda \mathfrak{F}(x(t))$.*

**5. Remarks concerning the application to existence proofs for eigen-
values of integral equations.** We consider the functional transformation

(4.8) $\mathfrak{F}(\mathfrak{x}) = \mathfrak{F}(x(t)) = \displaystyle\int_0^1 K(s, t) f(t, x(t)) dt$

where $K(s, t)$ is continuous in $0 \leqq s \leqq 1$, $0 \leqq t \leqq 1$, and the corresponding
eigen-value problem

(4.9) $\mathfrak{x} = \lambda \mathfrak{F}(\mathfrak{x})$.

As Birkhoff and Kellogg [11] have pointed out, the existence of a positive eigen-value with a corresponding positive eigen-function in case of a positive $K(s, t)$ is an immediate consequence of their general theorems if $f(t, x(t)) = x^2(t)$. But already if

(4.10) $f(t, x) = x + x^2$

their theory can not be applied. Indeed for $x(t) = -1$ we have $f = \mathfrak{F}(\mathfrak{x}) = 0$ and the condition $\| \mathfrak{F}(\mathfrak{x}) \| \geqq m > 0$ is not satisfied for *all* $\mathfrak{x}$ of the sphere $\| \mathfrak{x} \| = 1$ (cf. footnote 10 of the present paper). In view of this fact it might be worth mentioning that Theorem 4.1 of the present paper allows us to assert the existence of a positive eigen-value and a corresponding positive eigen-function of (4.8), (4.9) under the following conditions which contain (4.10) as special case: *there exist two positive numbers r and $\bar{m}$ and three non-negative functions $A(t)$, $B(t)$, and $C(t)$ of L^2, the last one being bounded,* such that for all $x(t) \subset L^2$ for which

$$\int_0^1 x^2(t)\,dt = r^2 \text{ and } x(t) \geqq 0 \text{ almost everywhere,}$$

$f(t, x(t))$ belongs to L^2 and the following inequalities hold:

$$K(s, t)f(t, x(t)) \geqq \bar{m}x^2(t) \qquad (\bar{m} > 0)$$

$$0 \leqq | f(t, x(t)) | \leqq A(t) + B(t)x(t) + C(t)x^2(t).$$

Finally we assume that the relation

$$\lim_{n \to \infty} \int_0^1 [x^n(t) - x(t)]^2\,dt = 0$$

implies

$$\lim_{n \to \infty} \int_0^1 [f(t, x^n(t)) - f(t, x(t))]^2\,dt = 0.$$

We omit the proof which consists of a simple verification of the hypotheses of Theorem 4.1.

To give also an example for Theorem 4.2 we replace the interval $0 \leqq t \leqq 1$ by the interval $-\pi \leqq t \leqq \pi$ and specify the system (4.6) by setting

(5.4) $\phi_0 = (1/\sqrt{2\pi})$, $\phi_{2n} = (1/\sqrt{\pi}) \cos nt$, $\phi_{2n-1} = (1/\sqrt{\pi}) \sin nt$,
                                                    $(n = 1, 2, \cdots)$

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[11] [2], p. 113.—For the following cf. footnote 10 of the present paper.

As can be shown [12] from Theorem 4.2 *the integral equation*

$$x(s) = \lambda \int_{-\pi}^{\pi} K(s-t)\{a(t) + b(t)x(t) + c(t)x^2(t)\}dt$$

has at least one positive eigen-value λ and a corresponding eigen-function $x(s)$ which is even and has non-negative components with respect to the system (5.4) if the following conditions are satisfied: $K(t)$ is a continuous function with the period 2π and $a(t)$, $b(t)$, $c(t)$ are bounded functions of L^2. All these functions are even and their components with respect to the system (5.4) are non-negative. Moreover, if a_n and K_n are the components of $a(t)$ and $K(t)$, the product $a_n K_n$ is different from zero for at least one value of n.

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[12] For the proof one has to notice that the product of two even functions with non-negative cosine components is not only even but has also non-negative cosine components.

HARMONIC CONTINUATION IN SPACE.[*][1]

By D. M. Seward.

1. The object of this paper is the extension to three-dimensional space of Hadamard's theorem [2] on harmonic continuation of harmonic functions in the plane. Our result is stated in Theorem I below. We use the following definitions.

(1.1) A set σ of points $P(x, y, z)$ in space is said to be an analytic surface set if, to each point p of σ, there corresponds a function $E_p(P) = E_p(x, y, z)$ which, in some sphere about p,[3] is analytic, has a non-vanishing gradient ∇E_p, and vanishes on, and only on, σ.

(1.2) A function $U(P)$, defined on a set σ of points in space is said to be analytic on σ if, to each point p of σ, there corresponds a function $V_p(P)$ which in some sphere about p, is analytic and coincides with $U(P)$ on σ.

THEOREM I. *Let D be a domain in space with boundary d; let the frontier of D contain an analytic surface set σ no point of which is at zero distance from $d - \sigma$; let $U(P) = U(x, y, z)$ be harmonic in D and let either*

(1.3) $U(P)$ *be continuous on $D + \sigma$ and analytic on σ, or*

(1.4) U, U_x, U_y, U_z *coincide in D with functions continuous on $D + \sigma$ and let $\partial U / \partial n$, the outer normal derivative of U on σ, be analytic on σ. Under the above hypotheses, U can be continued harmonically across σ. That is, there is a function $U^*(P)$, harmonic in a domain D^* containing $D + \sigma$, which coincides with $U(P)$ in D.*

2. We first reduce Theorem I to a "local theorem."

THEOREM II. *Let the hypotheses of Theorem I hold. Let O be a point of σ. Then $U(P)$ can be continued harmonically across σ at O. That is, there is a function $U_O(P)$, harmonic in a sphere R_O about O, such that $U_O(P) = U(P)$ in $D \cdot R_O$.*

That Theorem I is a consequence of Theorem II may be seen as follows. Assuming Theorem II, there corresponds to each point p of σ a sphere R_p about p and a function U_p, harmonic in R_p, which coincides with U in $D \cdot R_p$. Denote by R'_p the sphere about p with radius one third that of R_p. The set D^*, obtained by adjoining to D the sets R'_p for all p on σ, contains $D + \sigma$. Plainly D^* is open. Noting that D is a domain, that each R'_p is convex and contains points of D, we see that D^* is connected and therefore a domain.

Now define U^* in D^* as equal to U in D and equal to U_p in $R'_p - D \cdot R'_p$, for each p on σ. Then U^* is single valued. Otherwise there would be a point P_0 in two of the spheres R'_p and R'_q, with the radius of R'_p not smaller than that of R'_q, such that $U_p(P_0) \neq U_q(P_0)$. We should then have R'_q contained in R_p, and $U_p(P_0) = U(P_0) = U_q(P_0)$ in $D \cdot R'_q$. Since $D \cdot R'_q$ is a non-void open set, this would lead to a contradiction. Since U^* coincides with a harmonic function in the neighborhood of each point of D^*, U^* is harmonic in D^*. Thus Theorem I follows from Theorem II.

3. The rest of this paper is devoted to the proof of Theorem II. We develop the proof in a series of lemmas. We may suppose that the origin of our x-, y-, z-axes is at O, and that the positive or negative z-axis lies along the gradient at O of the function $E(P) = E(x, y, z)$ corresponding to O in the sense of definition (1.1), according as O is or is not a limit point of points of D on this gradient. With axes in the above position we have

$$E(o, o, o) = E_x(o, o, o) = E_y(o, o, o) = 0; \quad E_z(o, o, o) \neq 0.$$

Noting that there is a positive number h_0 such that, in the cube $|\,x\,|$, $|\,y\,|$, $|\,z\,| < h_0$, E is analytic and vanishes on and only on σ, we have, on applying results of implicit function theory [4]

LEMMA I. *There are positive numbers h_1 and h_2 less than h_0 and a function $f(x, y)$ such that $f(x, y)$ is analytic and $|\,f(x, y)\,| < h_1$ for $|\,x\,|$, $|\,y\,| < h_2$; the part σ_0 of σ inside the parallelepiped*

$$N: \quad |\,x\,|, \; |\,y\,| < h_2, \; |\,z\,| < h_1$$

is represented by $z = f(x, y)$, $|\,x\,|$, $|\,y\,| < h_2$; and

(3.1) $$f(0, 0) = f_x(0, 0) = f_y(0, 0) = 0.$$

The domain N, being within the cube $|\,x\,|$, $|\,y\,|$, $|\,z\,| < h_0$, contains no points of $D - \sigma$. Thus, since O is a frontier point of D, our choice of the positive z-axis results in

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[4] See, for example, E. Goursat, *Mathematical Analysis*, vol. 1 (1904), pp. 35 and 399.

LEMMA 2. *The sets N,*

$$N_1: \quad -h_1 < z < f(x,y), \ |x|, \ |y| < h_2, \ and$$
$$N_2: \quad f(x,y) < z < h_1, \ |x|, \ |y| < h_2$$

are domains such that $N = N_1 + N_2 + \sigma_0$, $D \cdot N_1 = N_1$, and $D \cdot N_2 = 0$.

4. We now define a domain $\triangle$ whose boundary is smooth enough to permit representation of U in $\triangle$ as the potential of a surface distribution.

LEMMA 3. *There can be constructed a domain $\triangle$ with boundary s such that* (a) *$\triangle + s$ is contained in $N_1 + \sigma$;* (b) *for a sufficiently small positive number h_3, $h_3 < h_2$, the part of s in the cylinder $r = \sqrt{x^2 + y^2} < h_3$, $|z| \leq h_3$, is represented by $z = f(x,y)$;* (c) *for each point p of s, there is a neighborhood of p, the portion of s within which, when referred to tangent-normal axes $(\lambda_1, \lambda_2, \lambda_3)$ at p, has a representation $\lambda_3 = \Lambda_p(\lambda_1, \lambda_2)$, the function Λ_p being one-valued and continuous with its first and second partial derivatives, for λ_1, λ_2 sufficiently small.*

There is a positive constant A_1 such that, for $|x|, |y| < h_2$, $|f(x,y)| \leq A_1 (x^2 + y^2) = A_1 r^2$. We choose h_3 so that $3h_3 < h_1$, $3h_3 < h_2$, and $9A_1 h_3 < 1$. Then $|f(x,y)| \leq A_1 r^2 \leq A_1 h_3^2$ for $r < 3h_3$. We set

$$g(r) = (2h_3 - r)^3 \cdot (4h_3^2 - 9h_3 r + 6r^2)/h_3^5.$$

The polynomial g has the properties:

$$g(h_3) = 1, \ g'(h_3) = g''(h_3) = g(2h_3) = g'(2h_3) = g''(2h_3) = 0.$$

We put

$$f^+(x,y) = \begin{cases} f(x,y) & r \leq h_3 \\ [f(x,y) + h_3]g(r) - h_3, & h_3 < r \leq 2h_3 \\ -2h_3 + \{h_3^4 - (r - 2h_3)^4\}^{1/4}, & 2h_3 < r \leq 3h_3 \end{cases}$$

$$f^-(x,y) = \begin{cases} -3h_3, & r \leq 2h_3 \\ -2h_3 - \{h_3^4 - (r - 2h_3)^4\}^{1/4}, & 2h_3 < r \leq 3h_3. \end{cases}$$

It can be verified that f^+ and f^- are continuous with their first and second partial derivatives.

For $\triangle$ we take the set of points for which

$$f^-(x,y) < z < f^+(x,y), \qquad r < 3h_3.$$

The boundary s of $\triangle$ consists of the surface $z = f^+(x,y)$, $z = f^-(x,y)$. Noting that the part of s for which $r \geq 2h_3$ coincides with the part of the surface

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of revolution $(z + 2h_3)^4 + (r - 2h_3)^4 = h_3^4$ for which $r \geqq 2h_3$, we see that $\triangle$ has the properties specified in (a), (b), and (c).

5. Domains bounded by surfaces having the tangent-normal representation property of (c), Lemma 3, are of the type considered by Kellogg [5] in his treatment of the Dirichlet and Neumann problems. Using Kellogg's results, we have

LEMMA 4. *Let* $K(p, q) = - 1/2\pi \; \partial/\partial n \; (1/pq)$, $p \neq q$, q *on* s, *and* n *being the outer normal to* s *at* q. *Referring to* (1.3) *of Theorem* I, *there is a function* $Z(p) = Z(x, y, z)$, *continuous on* s, *such that*

$$(5.1) \qquad Z(p) = - U(p) + \int \int_s K(p, q) Z(q) ds_q$$

for p *on* s; *and, for* P *in* $\triangle$,

$$(5.2) \qquad U(P) = \int \int_s K(P, q) Z(q) ds_q.$$

Referring to (1.4) *of Theorem* I, *there is a function* $Y(p) = Y(x, y, z$ *continuous on* s, *such that*

$$(5.3) \qquad Y(p) = \partial U/\partial n - \int \int_s Y(q) K(q, p) ds_q$$

for p *on* s; *and, for* P *in* $\triangle$,

$$(5.4) \qquad U(P) = (1/2\pi) \int \int_s Y(q) (1/Pq) ds_q.$$

6. With regard to the functions

$$Z(x, y) = Z(x, y, f(x, y)), \qquad Y(x, y) = Y(x, y, f(x, y))$$

of the above lemma, we have

LEMMA 5. *The conclusion of Theorem* II *holds if* $Z(x, y)$ *or* $Y(x, y)$ *is analytic at* $x = y = 0$.

We divide s into two parts:

$$s^*: \quad z = f(x, y), \; |x| < h_3/\sqrt{2}, \; |y| < h_3/\sqrt{2},$$

and $s - s^*$. Then s^* divides the parallelepiped N^*: $|x| < h_3/\sqrt{2}$, $|y| < h_3/\sqrt{2}$, $|z| < h_3$ into two domains; N^*_1: $|x| < h_3/\sqrt{2}$, $|y| < h_3/\sqrt{2}$, $-h_3 < z < f(x, y)$ and $N^*_2 = N^* - (N^*_1 + s^*)$, of which N^*_1 is in D and

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[5] O. D. Kellogg, *Foundations of Potential Theory*, Berlin (1929), Chap. XI. In particular, Theorem I, p. 212, Theorem I, p. 311, Theorem V, p. 314.

N^*_2 is exterior to D. By a theorem due to Schmidt,[6] if $Z(x, y)$ [or $Y(x, y)$] is analytic at $x = y = 0$, then there is a function $U^*_0(P)$, harmonic in a sphere R^*_0 about O which coincides with

$$\iint_{s^*} K(P, q)Z(q)\,ds_q \qquad [\text{or } (1/2\pi) \iint_{s^*} Y(q)/Pq\, ds_q]$$

in $R^*_0 \cdot N^*_1$. On the other hand, the function

$$U^{**}_0(P) = \iint_{s-s^*} K(P, q)Z(q)\,ds_q \qquad [\text{or } (1/2\pi) \iint_{s-s^*} Y(q)/Pq\, ds_q]$$

is harmonic at all points not on $s - s^*$. Choosing then a sufficiently small sphere R_0 about O and setting $U_0(P) = U^*_0(P) + U^{**}_0(P)$ for P in R_0, we conclude that the lemma is valid.

7. To complete the proof of Theorem II we have to show that $Z(x, y)$ [or $Y(x, y)$] is analytic at $x = y = 0$. Since the proof for Y is analogous to that for Z we confine the discussion to Z.

The function f has a power series expansion

$$f(x, y) = \sum_{\mu, \nu=0}^{\infty} a_{\mu\nu} x^\mu y^\nu$$

which, for sufficiently small x, y, $|x|$, $|y| < h_4$ say, converges absolutely and represents $f(x, y)$. We have $a_{10} = a_{01} = 0$. There are numbers A_2, h_5, $0 < h_5 < h_4$ such that the function

$$f^*(x, y) = \sum_{\mu, \nu=0}^{\infty} |a_{\mu\nu}|\, x^\mu y^\nu$$

has first partial derivatives less than 0.1 and second partial derivatives less than A_2 for $|x|$, $|y| < h_5$. We select arbitrarily $0 < k < h_5$, and denote by s_k the subset of s for which $z = f(x, y)$, $|x|$, $|y| < k$. The functions $V(x, y) = V(x, y, f(x, y))$ and $W(x, y) = W(x, y, f(x, y))$, where V is the function corresponding to O and U in the sense of definition (1.2), and

$$W(x, y, z) = \iint_{s-s_k} K(P, q)Z(q)\,ds_q,$$

are analytic at $x = y = 0$. We choose $h_6 > 0$ so that the expansions of $V(x, y)$ and $W(x, y)$ at $x = y = 0$ converge absolutely and represent these functions for $|x|$, $|y| < h_6$. We select h so that $0 < h < k$, $h < h_6$ and $h < 1/128\, A_2$. We put

[6] E. Schmidt, *Mathematische Annalen*, vol. 68 (1910), pp. 107-118. Also R. Bruns, *Journal für Mathematik*, vol. 81 (1876), pp. 349-356.

$$\phi(x, y, u, v) = \frac{(x-u)f_{10}(u, v) + (y-v)f_{01}(u, v) + f(u, v) - f(x, y)}{2\pi\{(x-u)^2 + (y-v)^2 + [f(u, v) - f(x, y)]^2\}^{3/2}}$$

for $|x|, |y|, |u|, |v| < k$. Then from (5.1) we have, for $|x|, |y| < h$,

$$(7.1) \qquad Z(x, y) = \sum_{j=1}^{j=4} \psi_j(x, y) + \int_{-h}^{h} du \int_{-h}^{h} \phi(x, y, u, v) Z(u, v) dv,$$

where

$$\psi_1(x, y) = 1/4[-V(x, y) + W(x, y)] + \int_{-k}^{-h} du \int_{-k}^{h} \phi(x, y, u, v) Z(u, v) dv,$$

and ψ_2, ψ_3, ψ_4 are similarly defined with the rectangle of integration $(-k, -k; -h, h)$ replaced in turn by $(-k, h; h, k)$, $(h, -h; k, k)$, $(-h, -k; k, -h)$. To prove that Z is analytic at $x = y = 0$ we shall show that there is a function $L(x, y)$, analytic, bounded and satisfying the equation

$$(7.2) \qquad L(x, y) = \psi_1(x, y) + \int_{-h}^{h} du \int_{-h}^{h} \phi(x, y, u, v) L(u, v) dv$$

for $|x|, |y| < h$. By symmetry and addition it then follows that there is a function $Z^*(x, y)$, analytic and bounded for $|x|, |y| < h$ such that (7.1) holds with Z^* in place of Z. The following uniqueness theorem serves to show that we must have $Z^* = Z$ for $|x|, |y| < h$.

LEMMA 6. *If $P(x, y)$ is continuous and bounded for $|x|, |y| < h$ and if*

$$P(x, y) = \int_{-h}^{h} du \int_{-h}^{h} \phi(x, y, u, v) P(u, v) dv$$

for $|x|, |y| < h$, then $P(x, y) \equiv 0$ for $|x|, |y| < h$.

The second partial derivatives of f are bounded by A_2 for $|x|, |y| < h$ so that

$$|\phi(x, y, u, v)| \leqq A_2[(x-u)^2 + (y-v)^2]^{-1/2}/2\pi$$

for $|x|, |y|, |u|, |v| < h$. Hence, for $|x|, |y| < h$

$$|P(x, y)| \leqq A_2 \| P \| /2\pi \int_{-h}^{h} du \int_{-h}^{h} [(x-u)^2 + (y-v)^2]^{-1/2} dv$$

$$\leqq A_2 \| P \| 2\sqrt{2}\, h$$

where $\| P \| = $ l. u. b. $| P(x, y) |$ for $|x|, |y| < h$. But, with $h < 1/128 A_2$, this can only hold if $\| P \| = 0$.

8. It was stated without conclusive proof by P. Lévy [7] that a function $Z(p)$ satisfying (5.1) might be proved analytic by allowing p to have com-

[7] P. Lévy, *Acta Mathematica*, vol. 42 (1920), pp. 232-235.

plex coördinates. We introduce complex variables $x = x' + ix''$, $y = y' + iy''$, $\cdots$, in which x', y', x'', y'', $\cdots$, are real. These are not, however, the coördinates suggested by Lévy.

Permitting x, y to assume complex values in the expansion of $f(x, y)$, we obtain a function $F(x, y)$, say, holomorphic in the domain $\delta_1 : |x|, |y| < h_5$. The first and second derivatives of F are bounded in absolute value by 0.1 and A_2 respectively. Similarly letting x, y assume complex values in the function $1/4(-V + W)$ we obtain a function $\Pi(x, y)$, holomorphic in the domain $\delta_2 : |x|, |y| < h_6$. We put

$$M_1(x, y, u, v) = (x - u) F_u(u, v) + (y - v) F_v(u, v) + F(u, v) - F(x, y),$$
$$M_2(x, y, u, v) = (x - u)^2 + (y - v)^2 + [F(u, v) - F(x, y)]^2.$$

These functions are holomorphic in the domain

$$\delta_3 : \quad |x|, |y|, |u|, |v| < h_5$$

in the x, y, u, v-space. We proceed with a series of lemmas.

LEMMA 7. *For (x, y, u, v) in δ_3 we have*

(8.1)
$$\begin{cases} |F(u, v) - F(x, y)|^2 \leq 0.1(|x - u|^2 + |y - v|^2), \\ |M_1(x, y, u, v)| \qquad \leq A_2(|x - u|^2 + |y - v|^2). \end{cases}$$

Supposing (x, y, u, v) to be in δ_3 and the integrals taken along linear paths, we have

$$|F(u, v) - F(x, y)|^2 = |\int_y^v F_{01}(u, s) ds + \int_x^u F_{10}(t, y) dt|^2$$
$$\leq 0.1(|x - u|^2 + |y - v|^2),$$

$$|M_1(x, y, u, v)| = |\int_u^x \{\int_t^u F_{20}(r, v) dr + \int_y^v F_{11}(t, s) ds\} dt$$
$$+ \int_v^y \{\int_s^v F_{02}(u, w) dw\} ds|$$
$$\leq A_2(|x - u|^2 + |y - v|^2).$$

LEMMA 8. *Let T denote the set of points in δ_3 for which*

(8.2) $(x'' - u'')^2 + (y'' - v'')^2 < 0.4[(x' - u')^2 + (y' - v')^2]$.

Then T is a domain and, for (x, y, u, v) in T, the real part $\mathfrak{R}[M_2(x, y, u, v)]$ of M_2 satisfies the inequality

(8.3) $\mathfrak{R}[M_2] > 0.4[(x' - u')^2 + (y' - v')^2] > 0$.

We see that T contains any real point (x', y', u', v') in δ_3 for which not both $x' = u'$ and $y' = v'$. In particular, the point $(-h, -h, h, h)$ is a point of T. The set T is clearly an open set. We now show that it is connected.

Let (X, Y, U, V) be a point of T and consider the points (x, y, u, v) of the set for which $x = X' + iX''(1-t)$, $0 \leq t \leq 1$, with like conditions on y, u, and v. These points lie in T, for (8.2) holds. As t varies from 0 to 1, the point (x, y, u, v) varies from (X, Y, U, V) to (X', Y', U', V'). Now either $X' \neq U'$ or $Y' \neq V'$. Suppose the former. Let y go from Y' to $-h$ and v from V' to $-h$, both through real values, holding $x = X'$ and $u = U'$. Now let x go to $-h$ and u to $-h$ along the real axis. The point (x, y, u, v) has gone from (X, Y, U, V) to $(-h, -h, h, h)$, remaining in T throughout. Thus T is a domain.

As for (8.3), we have, for (x, y, u, v) in T,

$$\mathcal{R}\,[M_2(x, y, u, v)] = (x' - u')^2 + (y' - v')^2 - (x'' - u'')^2 - (y'' - v'')^2$$
$$+ \mathcal{R}\,[f(u, v) - f(x, y)]^2$$
$$> 0.6[(x' - u')^2 + (y' - v')^2] - 0.1[\,|\,x - u\,|^2 + |\,y - v\,|^2]$$
$$= 0.5[(x' - u')^2 + (y' - v')^2] - 0.1[(x'' - u'')^2 + (y'' - v'')^2]$$
$$> 0.4[(x' - u')^2 + (y' - v')^2] > 0.$$

LEMMA 9.   *For (x, y, u, v) in T, we define*

$$\Phi(x, y, u, v) = \frac{M_1(x, y, u, v)}{2\pi\{J[M_2(x, y, u, v)]\}^3}$$

where

$$J(w) = \sqrt{|\,w\,|}\; e^{ia/2}, \quad w = |\,w\,|\,e^{ia}, \quad -\pi < a < \pi.$$

Then, for (x, y, u, v) in T, $\Phi(x, y, u, v)$ is holomorphic and

$$(8.4)\qquad |\,\Phi(x, y, u, v)\,| \leq \frac{2A_2}{\sqrt{(x' - u')^2 + (y' - v')^2}}\;.$$

For real (x, y, u, v) in T, $\Phi(x, y, u, v)$ reduces to $\phi(x, y, u, v)$. The functions M_1 and M_2 are holomorphic in T and $\mathcal{R}\,[M_2] > 0$. $J(w)$ is holomorphic in the right half-plane, $\mathcal{R}\,[w] > 0$, so that $J(M_2)$ and $[J(M_2)]^3$ are holomorphic in T. Since $J(M_2) \neq 0$ in T, it follows that $\Phi(x, y, u, v)$ is holomorphic in T.

To get the bound (8.4) we make use of (8.1), (8.2), and (8.3) getting

$$|\,\Phi(x, y, u, v)\,| = \frac{|\,M_1(x, y, u, v)\,|}{2\pi\,|\,M_2(x, y, u, v)\,|^{3/2}} < \frac{A_2(|\,x - u\,|^2 + |\,y - v\,|^2)}{2\pi[\mathcal{R}\,(M_2)]^{3/2}}$$
$$\leq \frac{A_2[(x' - u')^2 + (y' - v')^2 + (x'' - u'')^2 + (y'' - v'')^2]}{0.4\pi[(x' - u')^2 + (y' - v')^2]^{3/2}}$$
$$\leq \frac{1.4A_2[(x' - u')^2 + (y' - v')^2]}{0.4\pi[(x' - u')^2 + (y' - v')^2]^{3/2}} \leq \frac{2A_2}{\sqrt{(x' - u')^2 + (y' - v')^2}}$$

LEMMA 10. *Let* $H = H(x, y)$ *be the set of points in the* (x, y)-*space for which* $|x''| < 0.4(h - |x'|)$, $|y''| < 0.4(h - |y'|)$, *and* $|y''| < 0.4(h + x')$. *Then H is a domain and the function*

$$(8.5) \quad \Psi_1(x, y) = \Pi(x, y) + \int_{-k}^{-h} du' \int_{-k}^{h} \Phi(x, y, u', v') Z(u', v') dv'$$

is holomorphic and bounded in H.

For real (x, y) in H, $\Psi_1(x, y)$ reduces to ψ_1 of **7**. That H is a domain can be proved in the same way that T was shown to be a domain.

We verify that when (x, y) is in H, $u = u'$, $v = v'$, $-k \leq u' \leq -h$, $-k \leq v' \leq h$, we have $|x|, |y|, |u|, |v| \leq k < h_5$ and

$$[(x'' - u'')^2 + (y'' - v'')^2]^{1/2} = [(x'')^2 + (y'')^2]^{1/2}$$
$$< (0.32)^{1/2}(h + x') < (0.32)^{1/2}(x' - u').$$

We conclude that (x, y, u, v) lies in T. Since $\Phi(x, y, u, v) Z(u, v)$ is continuous for (x, y) in H, $-k \leq u' \leq -h$, $-k \leq v' \leq h$, and for each such (u', v') is holomorphic in H, it follows that the integral in (8.5) represents a function holomorphic in H. On applying (8.4) we see that this integral is bounded for (x, y) in H. The function $\Pi(x, y)$ being holomorphic in δ_2 which contains H, we conclude that the lemma is true.

LEMMA 11. *Let* $G(x, y)$ *be holomorphic and bounded in H. Let* H_1 *be the set of points in the* (x, y, u)-*space for which*

$$H_1: \quad (x, y) \text{ is in } H, \ |u''| < 0.4(u' + h), \ |u'' - x''| < 0.4(x' - u').$$

For (x, y, u) *in* H_1, *let* $C(x, y, u)$ *be the open polygonal path in the v-plane from* $-h$ *to* V^- *to* V^+ *to* h, *where*

$$V^- = V^-(x, y, u) = -h + \frac{u' + h}{x' + h}(y + h), \qquad V^+ = h + \frac{u' + h}{x' + h}(y - h).$$

Then H_1 *is a domain and the function*

$$(8.6) \qquad\qquad B(x, y, u) = \int_{C(x, y, u)} \Phi(x, y, u, v) dv$$

is holomorphic in H_1.

That H_1 is a domain can be proved in the same way that T was shown to be a domain.

Let (x, y, u) be in H_1 and v on C. Then we have $|x|, |y|, |u|, |v| < h$, $|x'' - u''| < 0.4(x' - u')$ and either $|y'' - v''| < 0.4(y' - v')$ or

$| y'' - v'' | < | y'' | (x' - u')/(x' + h) < 0.4(x' - u')$ or $| y'' - v'' |$
$< 0.4(v' - y')$.   Furthermore, we have

$$| u'' | < 0.4(h - | u' |), \quad | v'' | < 0.4(h - | v' |)$$

and

$$| v'' | < | y'' | \left| \frac{u' + h}{x' + h} \right| < 0.4(u' + h).$$

It follows that (x, y, u, v) is in T and that (u, v) is in $H(u, v)$. Using the inequality (8.4), we see that the integral (8.6) exists.

To prove that B is holomorphic in H_1 it is sufficient to show that, if E is any domain whatever which with its closure lies in H_1, B is holomorphic in E. Let E be any such set, now fixed. For (x, y, u) in E, $(x' + h)/(u' + h)$ is bounded above by an integer m. For $n \geq m + 1$, the points

$$V_n^1(y) = - h + (y + h)/n, \qquad V_n^2(y) = h + (y - h)/n$$

lie respectively on the left- and right-hand segments of C. For $n \geq m + 1$, we denote by $C_n = C_n(x, y, u)$ the polygonal path in the v-plane from V_n^1 to V^- to V^+ to V_n^2. We put

$$B^n(x, y, u) = \int_{C_n} \Phi(x, y, u, v) G(u, v) dv.$$

We find by use of (8.4) that $| B^n |$ does not exceed a constant multiple of $\log 6h/(x' - u')$. Accordingly the functions B^n are uniformly bounded in E. Furthermore, for a fixed (x, y, u) in E, the limit as $n \to \infty$ of B^n is B. To prove that B is holomorphic in H_1, it suffices, as a consequence of Montel's theorem on uniformly bounded sequences of holomorphic functions,[8] to show that B^n is holomorphic in E for all $n \geq m + 1$. To establish this fact, we shall show that B^n is differentiable with respect to x, y and u in E.

Let (x_0, y_0, u_0) be a point of E. The set of points (x_0, y_0, u_0, v), v on C_n, is a closed subset of T. The set of points (u_0, v), v on C_n, is a closed subset of $H(u, v)$. Hence there is a positive constant δ such that, if $| \Delta x |, | \Delta y |, | \Delta u |, | \Delta v | < \delta$ and v is on C_n, then $(x_0 + \Delta x, y_0 + \Delta y, u_0 + \Delta u, v + \Delta v)$ is in T and $(u_0 + \Delta u, v + \Delta v)$ is in $H(u, v)$. We note that the set $Q(v)$: v on C_n, $| \Delta v | < \delta$, of points $v + \Delta v$ in the v-plane is a simply connected domain.

We let

$$\Delta_x B^n = B^n(x_0 + \Delta x, y_0, u_0) - B^n(x_0, y_0, u_0).$$

For $0 < | \Delta x |$ and sufficiently small, the path $C_n(x_0 + \Delta x, y_0, u_0)$ lies in $Q(v)$.

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[8] P. Montel, *Leçons sur les familles normales de fonctions analytiques*, Paris (1927), p. 241.

For Δx fixed and $0 < |\Delta x| < \delta$, $\Phi(x_0 + \Delta x, y_0, u_0, v) G(u_0, v)$ is a holomorphic function of v in $Q(v)$. The end points of $C_n(x_0 + \Delta x, y_0, u_0)$ and $C_n(x_0, y_0, u_0)$ are the same. Consequently we can apply Cauchy's theorem. We obtain

$$\Delta_x B^n = \int_{C_n} [\Phi(x_0 + \Delta x, y_0, u_0, v) - \Phi(x_0, y_0, u_0, v)] G(u_0, v) dv.$$

Now, for v on the closed set $C_n(x_0, y_0, u_0)$, the difference quotient of Φ with respect to x tends uniformly to $\Phi_x(x_0, y_0, u_0, v)$ as $\Delta x \to 0$. Thus, since $G(u_0, v)$ is bounded on C_n, we conclude that B_x^n exists. The proof that B_u^n exists is similar to the foregoing and will be omitted.

It remains to consider differentiability with respect to y. We let

$$\Delta_y B^n = B^n(x_0, y_0 + \Delta y, u_0) - B^n(x_0, y_0, u_0).$$

If $|\Delta y|$ is sufficiently small, the curve $C_n(x_0, y_0 + \Delta y, u_0)$ lies in $Q(v)$. We change the path in the first term of $\Delta_y B^n$, getting

$$\Delta_y B^n = \int_{C_n(x_0, y_0, u_0)} [\Phi(x_0, y_0 + \Delta y, u_0, v) - \Phi(x_0, y_0, u_0, v)] dv$$
$$+ \left\{ \int_{V_n^1(y_0 + \Delta y)}^{V_n^1(y_0)} + \int_{V_n^2(y_0 + \Delta y)}^{V_n^2(y_0)} \right\} \Phi(x_0, y_0 + \Delta y, u_0, v) G(u_0, v) dv.$$

We now divide by Δy and let $\Delta y \to 0$. As in the case of $\Delta_x B^n$ the limit of the first term exists. Noting that

$$\lim_{\substack{\Delta y \to 0 \\ v \to v_0}} \Phi(x_0, y_0 + \Delta y, u_0, v) G(u_0, v) = \Phi G |_{x_0, y_0, u_0, v_0}$$

and that

$$V_n^j(y_0 + \Delta y) = V_n^j(y_0) + \Delta y / n, \qquad (j = 1, 2),$$

we find that B_y^n exists.

LEMMA 12. *Let G and B denote the functions defined in Lemma 11. For (x, y) in H, let $\Gamma(x)$ denote the open linear path in the u-plane from $-h$ to x. Then*

$$(8.8) \qquad B^*(x, y) = \int_{\Gamma(x)} B(x, y, u) du$$

is holomorphic in H and

$$(8.9) \qquad |B^*| \leqq 64 A_2 \|G\|_H h,$$

where $\|G\|_H = $ l. u. b. $|G|$ for (x, y) in H.

For (x, y) in H and u on $\Gamma(x)$, we have $|u''| < 0.4(u' + h)$, $|u'' - x''| < 0.4(x' - u')$. Hence (x, y, u) lies in H_1. We obtain the existence of the integral (8.8) and the bound (8.9) on noting that, for (x, y) in H,

$$\int_{\Gamma(x)} |\,du\,|\int_{C(x,y,u)} |\,\Phi(x,y,u,v)\,|\ |\,G(u,v)\,|\ |\,dv\,| \leqq 64\,A_2\,\|\,G\,\|_H\,h.$$

Now let $\Gamma_n(x)$ be the linear path in the u-plane from $V_n^1(x)$ to $V_N^1(x)$ where $N = n/(n-1)$. For $n > 2$, let $B^*_n(x,y)$ be the integral of $B(x,y,u)$ over $\Gamma_n(x)$. For (x,y) in H we have

$$|\,B^*_n(x,y)\,| \leqq |\,B^*(x,y)\,| \leqq 64\,A_2\,\|\,G\,\|_H\,h.$$

As $n \to \infty$, $B^*_n(x,y) \to B^*(x,y)$. To show that B^* is holomorphic in H it suffices to show that B^*_n is differentiable in H. Let (x_0, y_0) be a point of H. Using the methods of Lemma 11, we find that B^* in differentiable with respect to x and y at (x_0, y_0).

LEMMA 13. *Let $G(x,y)$ be holomorphic and bounded in H. Let H_2 be the set of points in the (x,y,u)-space for which*

$$H_2: \quad (x,y) \text{ is in } H, \; |\,u''\,| < 0.4(h-u'), \; |\,u''-x''\,| < 0.4(u'-x').$$

For (x,y,u) in H_2, let $C(y)$ be the open polygonal path in the v-plane from $-h$ to y to h. Then H_2 is a domain and the function

$$b(x,y,u) = \int_{C(y)} \Phi(x,y,u,v)\,G(u,v)\,dv$$

is holomorphic in H_2. For (x,y) in H, let $\Gamma^+(x)$ denote the open linear path in the u-plane from x to h. In H, the function

$$b^*(x,y) = \int_{\Gamma^+(x)} b(x,y,u)\,du$$

is holomorphic and

$$|\,b^*\,| \leqq 64\,A_2\,\|\,G\,\|_H\,h.$$

The reasoning by which this lemma is proved is similar to that for Lemmas 11 and 12 and may be omitted. In Lemma 11 we permitted the path $C(x,y,u)$ to get only as far from the axis of reals in the v-plane as $(u'+h)\,|\,y''\,|/(x'+h)$ in order to obtain the inequality

$$|\,v''\,| \leqq |\,y''\,|\,(u'+h)/(x'+h) < 0.4(x'+h).$$

In the present case, we have $x' < u'$ and thus

$$|\,v''\,| \leqq |\,y''\,| \leqq |\,y''\,|\,(u'+h)/(x'+h) < 0.4(x'+h).$$

LEMMA 14. *Let $G(x,y)$ be holomorphic and bounded in H. For (x,y) in H the function*

$$G^*(x,y) = \int_{C(x)} du \int_{C^*} \Phi(x,y,u,v)\,G(u,v)\,dv$$

where

$$C^* = \begin{cases} C(x, y, u) & \text{if } u' < x' \\ C(y) & \text{if } x' < u' \end{cases}$$

is holomorphic and

$$| G^* | \leqq 128 A_2 \| G \|_H h.$$

9. LEMMA 15. *There exists a function $l(x, y)$, holomorphic and bounded in H, such that*

$$l(x, y) = \Psi_1(x, y) + \int_{C(x)} du \int_{C^*} \Phi(x, y, u, v) l(u, v) dv$$

for (x, y) in H.

By Lemma 10 Ψ_1 is holomorphic and bounded by $\| \Psi_1 \|_H$. We put

$$l_1(x, y) = \int_{C(x)} du \int_{C^*} \Phi(x, y, u, v) \Psi_1(u, v) dv$$

$$l_n(x, y) = \int_{C(x)} du \int_{C^*} \Phi(x, y, u, v) l_{n-1}(u, v) dv, \qquad (n = 2, 3, \cdots).$$

These functions are holomorphic in H and

$$\| l_n(x, y) \|_H \leqq (128 A_2 h)^n \| \Psi_1 \|_H.$$

As $(128 A_2 h) < 1$, it follows that the series $\sum_1^\infty l_n(x, y)$ converges uniformly in H and defines there the required function $l(x, y)$.

LEMMA 16. *Let x, y be real. There exists a function $L(x, y)$, analytic and bounded for $|x|, |y| < h$, such that equation (7.2) holds for $|x|, |y| < h$.*

Supposing x, y real, $|x|, |y| < h$, then (x, y) as a complex point lies in H. Thus $L(x, y) = l(x, y)$ is analytic and bounded for $|x|, |y| < h$. In addition,

$$\int_{C(x)} du \int_{C^*} \Phi(x, y, u, v) l(u, v) dv = \int_{-h}^h du \int_{-h}^h \phi(x, y, u, v) L(u, v) dv,$$

$$\Psi_1(x, y) = \psi_1(x, y)$$

so that (7.2) holds. This lemma validates the assertions of **7** from which follow the truth of Theorems I and II.

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ON THE MOVEMENT OF A COSMIC CLOUD.*

By A. Rosenblatt.

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1. I presented in 1926 a Note to the *Accademia Nazionale dei Lincei*: " Sur le cas de la collision générale dans le problème des trois corps " (Vol. 3, Ser. vi (1926)) in which I gave the canonical form of the equations in the case of a triple collision with a simple singularity in the point of collision. At the same time I began the study of the movement of a cloud of cosmic dust of finite dimensions subjected only to Newton's law of attraction. The density ρ was supposed to be a function of the coördinates x, y, z and the time t which is integrable in Riemann's sense. I proved the analogon of Sundman's first theorem.

THEOREM 1. *" A cosmic cloud cannot tend to its center of mass, its momentum of inertia tending to zero, if the vector **K** of areas is positive."*

This result has never been published. Recently Professor G. Garcia obtained, by a very interesting method, Sundman's inequality in the case of n bodies. He passed then to the case of a cosmic cloud giving the generalization of Sundman's inequality for this cloud.

I have revised my old results and have succeeded in deducing Professor Garcia's beautiful result by my method, using Schwarz's inequality for integrals. I venture to publish these results adding some considerations which I believe to be new.

2. I suppose the initial state to be given by the velocity distribution

$$(1) \quad u_0 = u(x_0, y_0, z_0, t_0), \quad v_0 = v(x_0, y_0, z_0, t_0), \quad w_0 = w(x_0, y_0, z_0, t_0).$$

We have the condition of the invariance of mass

$$(2) \qquad (\rho/\rho_0)\, D(x, y, z)/D(x_0, y_0, z_0) = 1,$$

or

$$(3) \qquad \rho dv = \rho_0 dv_0,$$

$$(4) \qquad M = \int_{V_0} \rho_0 dv_0 = \int_V \rho dv.$$

The equations of motion are

$$(5) \qquad du/dt = \partial U/\partial x, \quad dv/dt = \partial U/\partial y, \quad dw/dt = \partial U/\partial z,$$

$$(6) \qquad U = \int_{V'} \rho' dv'/d = \int_{V'_0} \rho'_0 dv'_0/d,$$

$$(7) \qquad d = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}.$$

The autopotential W is

$$(8) \qquad W = \tfrac{1}{2} \int_V \int_{V'} \rho\rho' dv dv'/d = \tfrac{1}{2} \int_{V_0} \int_{V'_0} \rho_0 \rho'_0 dv_0 dv'_0/d.$$

Introducing the function

$$(9) \qquad U' = \int_V \rho dv/d = \int_{V_0} \rho_0 dv_0/d$$

and setting

$$f^2 = u^2 + v^2 + w^2, \qquad f' = u'^2 + v'^2 + w'^2$$

we have

$$\tfrac{1}{2}(d/dt)f^2 = u(du/dt) + v(dv/dt) + w(dw/dt),$$

$$\tfrac{1}{2}(d/dt)f'^2 = u'(du'/dt) + v'(dv'/dt) + w'(dw'/dt),$$

$$(d/dt)\{\tfrac{1}{2}\int \rho f^2 dv + \tfrac{1}{2}\int \rho' f'^2 dv'\} = (d/dt)\{\int \rho dv(u\partial U/\partial x + v\partial U/\partial y + w\partial U/\partial z)$$

$$+ \int \rho' dv'(u'\partial U'/\partial x' + v'\partial U'/\partial y' + w'\partial U'/\partial z')\}$$

$$= (d/dt)\{\int\int \rho dv \rho' dv' \cdot (u(\partial d^{-1}/\partial x) + v(\partial d^{-1}/\partial y) + w(\partial d^{-1}/\partial z)$$

$$+ \int\int \rho dv \rho' dv'(u'(\partial d^{-1}/\partial x') + v'(\partial d^{-1}/\partial y') + w'(\partial d^{-1}/\partial z'))\}$$

$$= \int\int \rho\rho' dv dv'(d/dt)(1/d) = 2(dW/dt).$$

Putting

$$T = \tfrac{1}{2}\int_V \rho f^2 dv$$

we have the equation of energy

$$(10) \qquad T = W + C.$$

3. Let us consider the moment of inertia I

$$(11) \qquad I = \int \rho dv r^2 = \int \rho_0 dv_0 r^2, \qquad r^2 = x^2 + y^2 + z^2.$$

We have

$$MI = \int (\sqrt{\rho dv})^2 \int (\sqrt{\rho dv} \cdot r)^2 = J^2 + \tfrac{1}{2}\int_V \int_{V'} \rho dv \rho' dv' \cdot (r - r')^2,$$

where we have put

$$(12) \qquad\qquad J = \int \rho r dv.$$

We have

$$\int \rho dv (du/dt) = (d^2/dt^2) \int \rho x dv = \int \int \rho dv \rho' dv' (\partial d^{-1}/\partial x)$$
$$= -\int \int \rho dv \rho' dv' [(x-x')/d^3] = 0,$$

and denoting by ξ, η, ζ the coördinates of the center of gravity

$$(13) \qquad \int \rho dv \cdot x = M\xi, \qquad \int \rho dv \cdot y = M\eta, \qquad \int \rho dv \cdot z = M\zeta$$

we have

$$d^2\xi/dt^2 = d^2\eta/dt^2 = d^2\zeta/dt^2 = 0,$$

so that we suppose

$$(14) \qquad\qquad \int \rho dv \cdot x = \int \rho dv \cdot y = \int \rho dv \cdot z = 0,$$

$$(15) \qquad\qquad \int \rho dv \cdot u = \int \rho dv \cdot v = \int \rho dv \cdot w = 0.$$

We have, therefore,

$$M \int \rho dv \cdot x^2 = \int (\sqrt{\rho dv})^2 \cdot \int (\sqrt{\rho dv} \cdot x)^2$$
$$= (\int \rho dv \cdot x)^2 + \tfrac{1}{2} \int \int \rho \rho' dv dv' (x-x')^2 \text{ etc.,}$$

so that we obtain the formula

$$(16) \qquad\qquad MI = \tfrac{1}{2} \int \int \rho \rho' dv dv' \cdot d^2.$$

We have also

$$M \int \rho dv \cdot u^2 = (\int \rho dv \cdot u)^2 + \tfrac{1}{2} \int \int \rho dv \rho' dv' (u-u')^2, \text{ etc.,}$$

so that

$$(17) \quad MT = \tfrac{1}{4} \int \int \rho dv \rho' dv' [(u-u')^2 + (v-v')^2 + (w-w')^2].$$

4. We have

$$(d/dt) \int \rho dv (yz' - zy') = \int \rho dv (y\partial U/\partial z - z\partial U/\partial y)$$
$$= \int \int \rho dv \rho' dv' \{-- y(z-z')/d^3 + z(y-y')/d^3\}$$
$$= \int \int \rho dv \rho' dv' \cdot (yz' - zy')/d^3 = 0,$$

so that we get the 3 integrals of areas

(18)
$$\int \rho dv(yz' - zy') = K_x,$$

$$\int \rho dv(zx' - xz') = K_y,$$

$$\int \rho dv(xy' - yx') = K_z.$$

5. We have

$$2IT = \int \rho dv r^2 \cdot \int \rho dv \cdot f^2 = (\int \rho dv fr)^2 + \tfrac{1}{2} \int \int \rho dv \rho' dv' (rV' - r'V)^2,$$

$$r^2 f^2 = (xx' + yy' + zz')^2 + (xy' - x'y)^2 + (yz' - y'z)^2 + (zx' - z'x)^2,$$

$$rf \gtreqless \sqrt{a^2 + b^2 + c^2},$$

$a = yz' - y'z,\ b = zx' - z'x,\ c = xy' - x'y.$ Further

$$(\sum m_i \sqrt{a_i^2 + b_i^2 + c_i^2})^2 = \sum m_i^2 (a_i^2 + b_i^2 + c_i^2)$$
$$+ \sum_{i \neq j} m_i m_j \sqrt{a_i^2 + b_i^2 + c_i^2} \cdot \sqrt{a_j^2 + b_j^2 + c_j^2} = (\sum m_i a_i)^2 + (\sum m_i b_i)^2$$
$$+ (\sum m_i c_i)^2 + \sum_{i \neq j} m_i m_j \{ \sqrt{a_i^2 + b_i^2 + c_i^2} \sqrt{a_j^2 + b_j^2 + c_j^2}$$
$$- a_i a_j - b_i b_j - c_i c_j \}.$$

Also

$$(a_i^2 + b_i^2 + c_i^2)(a_j^2 + b_j^2 + c_j^2) = (a_i a_j + b_i b_j + c_i c_j)^2$$
$$+ (a_i b_j - a_j b_i)^2 + (b_j c_i - b_i c_j)^2 + (c_j a_i - c_i a_j)^2,$$

so that

$$(\sum m_i \sqrt{a_i^2 + b_i^2 + c_i^2})^2 \geqq (\sum m_i a_i)^2 + (\sum m_i b_i)^2 + (\sum m_i c_i)^2,$$

$$(\int \rho dv rf)^2 \geqq (\int \rho dv a)^2 + (\int \rho dv \cdot b)^2 + (\int \rho dv \cdot c)^2$$
$$= K_x^2 + K_y^2 + K_z^2.$$

Hence

$$(\int \rho dv \sqrt{a^2 + b^2 + c^2})^2 = (\int \rho dv \cdot a)^2 + (\int \rho dv b)^2 + (\int \rho dv \cdot c)^2$$
$$+ \tfrac{1}{2} \int \int \rho dv \rho' dv' \{ \sqrt{a^2 + b^2 + c^2} \sqrt{a'^2 + b'^2 + c'^2} - (aa' + bb' + cc') \},$$

$$\int \rho dv \cdot rf = \int \rho dv [\sqrt{r^2 r'^2 + a^2 + b^2 + c^2} - \sqrt{a^2 + b^2 + c^2}] + \int \rho dv \sqrt{a^2 + b^2 + c^2},$$

$$2IT = \tfrac{1}{2} \int \int \rho dv \rho' dv' (rV' - r'V)^2$$

$$+ \{ \int \rho dv \sqrt{a^2 + b^2 + c^2} + \int \rho dv [\sqrt{r^2 r'^2 + a^2 + b^2 + c^2} - \sqrt{a^2 + b^2 + c^2}]^2.$$

Thus we obtain the inequality

(19) $$2IT \geqq K^2,$$

where K is the magnitude of the vector of areas.

6. Let us now suppose that I tends to zero as t tends to t_1 (finite). The mass $\bar{m}$ exterior to a sphere of radius $r \geqq R$ would then satisfy the inequality

$$\bar{m}R^2 \leqq I < \epsilon,$$

ϵ being > 0 and arbitrarily small,

$$\bar{m} = \int \rho_0 dv_0 < \epsilon/R^2$$

the integral being extended over the domain $r \geqq R$.

The mass in the sphere of radius r is

$$m_r = \int \rho_0 dv_0 - \bar{m} > M - \epsilon/R^2.$$

It follows that W tends to $+ \infty$ if $t \to t_1$. Indeed

$$W = \tfrac{1}{2} \int \int (\rho dv \rho' dv'/d) \geqq (1/4R) \int \int \rho dv \rho' dv' > (1/4R)(M - \epsilon/R^2)^2.$$

Choosing $R < \eta$, and then $\epsilon/R^2 < \zeta$, ζ arbitrarily small, we would have

(20) $$W \geqq (1/4\eta)(M - \zeta)^2.$$

It follows that if $I \to 0$ then $W \to + \infty$, $I'' \to + \infty$, I' grows constantly being negative, otherwise I would not tend to zero for $t \to t_1$, and I diminishes constantly to zero.

If $I \to 0$, I' is always negative and I can be taken as the independent variable. We have

$$dI'/dt = (dI'/dI)I' = \tfrac{1}{2}(d/dI)(I'^2) = 2W + 4C = 2T + 2C,$$

(21) $$I'^2 - I_0'^2 = 4 \int_{I_0}^{I} TdI + 4C \int_{I_0}^{I} dI.$$

Hence

$$\int_{I_0}^{I} TdI,$$

tends to a finite limit, which is negative, as $t \to t_0$. It follows from (19) that we have

$$K^2 \int_{I_0}^{I} (dI/2I) \geqq \int_{I_0}^{I} TdI$$

and $\frac{1}{2}K^2(\log I - \log I_0)$ tends to a finite negative limit as $I \to 0$, which is only possible if $K = 0$. So we have Sundman's Theorem 1, a result obtained, as mentioned, in 1926, but never published.

7. We shall now show that we obtain Sundman's inequality generalized in Professor G. Garcia's remarkable papers. To this end we proceed as follows.

We have

$$I = \int \rho dv r^2, \qquad I' = 2 \int \rho dv (xx' + yy' + zz'),$$

$$I'' = 2 \int \rho dv (xx'' + yy'' + zz'' + x'^2 + y'^2 + z'^2)$$

$$= 2 \int \rho dv f^2 + 2 \int \rho dv (x\partial U/\partial x + y\partial U/\partial y + z\partial U/\partial z).$$

$$\int \rho dv (x\partial U/\partial x + y\partial U/\partial y + z\partial U/\partial z) + \int \rho' dv' (x'\partial U'/\partial x' + y'\partial U'/\partial y' + z'\partial U'/\partial z')$$

$$= -\int\int \rho\rho' dv dv' \cdot 1/d = -2W = 2\rho dv (x\partial U/\partial x + y\partial U/\partial y + z\partial U/\partial z).$$

From these we obtain the relation of Lagrange

$$(22) \qquad I'' = 4T - 2W = 2W + 4C = 2T + 2C.$$

8. We can transform the expression for the kinetic energy. We have, on denoting by S the sum of the 3 components,

$$r^2 S x'^2 = (S x x')^2 + S(xy' - x'y)^2,$$

$$S x'^2 = r'^2 + S(xy' - x'y)^2/r^2,$$

$$2T = \int \rho dv \cdot r'^2 + \int \rho dv [S(xy' - x'y)^2/r^2].$$

We have

$$\int \rho dv \cdot r^2 \int \rho dv \cdot r'^2 = \int (\sqrt{\rho dv} \cdot r)^2 \cdot \int (\sqrt{\rho dv} \cdot r')^2$$

$$= (\int \rho dv r r')^2 - \int\int \rho_1 \rho_2 dv_1 dv_2 r_1 r'_1 r_2 r'_2 + \frac{1}{2} \int\int \rho_1 \rho_2 dv_1 dv_2 (r_1^2 r'_2{}^2 + r_2^2 r'_1{}^2)$$

$$= (\int \rho dv r r')^2 + \frac{1}{2} \int\int \rho_1 \rho_2 dv_1 dv_2 (r_1 r'_2 - r_2 r'_1)^2,$$

$$\int \rho dv r'^2 = I'^2/4I + (1/2I) \int\int \rho_1 \rho_2 dv_1 dv_2 (r_1 r'_2 - r_2 r'_1)^2,$$

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(23) $\quad I'^2/4I + (1/2I) \displaystyle\iint \rho_1\rho_2 dv_1 dv_2 (r_1 r'_2 - r_2 r'_1)^2$

$$+ \int \rho dv [\mathbf{S}(xy' - x'y)^2/r^2] = 2W + 2C = I'' - 2C.$$

9. We have

$$\int \rho dv r^2 \cdot \int \rho dv [\mathbf{S}(xy' - x'y)^2/r^2] = \mathbf{S}\Big(\int \rho dv (xy' - x'y)\Big)^2$$

$$+ \tfrac{1}{2}\mathbf{S}\iint \rho_1\rho_2 dv_1 dv_2 [(r_1/r_2)(xy' - x'y)_2 - (r_2/r_1)(xy' - x'y')_1]^2.$$

Thus we have obtained Sundman's relation generalized by Professor Garcia

(24) $\quad I'' - 2C = I'^2/4I + (1/2I) \displaystyle\iint \rho_1 dv_1 \rho_2 dv_2 (r_1 r'_2 - r_2 r'_1)^2$

$$+ K^2/I + (1/2I)\mathbf{S}\iint \rho_1\rho_2 dv_1 dv_2$$

$$\cdot [(r_1/r_2)(yz' - zy')_2 - (r_2/r_1)(yz' - zy')_1]^2,$$

and putting $I = R^2$ and, following Professor Birkhoff and Professor Garcia,

(25) $$H = RR'^2 - 2CR + K^2/R^2$$

we have

$$2RR'' + R'^2 - K^2/R^2 - 2C \gqq 0,$$

(26) $\quad H' = R'\{2RR'' + R'^2 - 2C - K^2/R^2\} = FR', \quad F \gqq 0,$

which is Professor Garcia's inequality.

10. Multiplying (23) by $I'/\sqrt{I}$ and integrating we get

$$\tfrac{1}{2}\, d/dt\ I'^2/\sqrt{I} = I'I''/\sqrt{I} - I'^3/4I^{3/2} = -\, d/dt\ 2K^2/\sqrt{I}$$

$$+ C\sqrt{I}' + I'/2I^{3/2}\Big\{\iint \rho_1\rho_2 dv_1 dv_2 (r_1 r'_2 - r_2 r'_1)^2 + SQ_x^2\Big\},$$

$$Q_x^2 = \iint \rho_1\rho_2 dv_1 dv_2 [(r_1/r_2)(yz' - zy')_2 - (r_2/r_1)(yz' - zy')_1]^2 \text{ etc.,}$$

(27) $\quad I'^2/2\sqrt{I} + 2K^2/\sqrt{I} - 4C\sqrt{I} = \displaystyle\int (I'/2I^{3/2})\{SQ_x^2$

$$+ \iint \rho_1\rho_2 dv_1 dv_2 (r_1 r'_2 - r_2 r'_1)^2\}dt + \text{Const.}$$

It may be also remarked that (19) is an immediate consequence of (24).

11. Let us suppose that I tends to zero, K being zero. We have from (24)

$$I'' - 2C \geqq I'^2/4I, \qquad\qquad \tfrac{1}{2}(d/dt)(I'^2) - 2CI' \leqq I'^3/4I,$$

$$\tfrac{1}{2}(d/dI)(I'^2)I' - 2CI' \leqq I'^3/4I, \qquad \tfrac{1}{2}(d/dI)(I'^2) - 2C \geqq I'^2/4I,$$

$$\tfrac{1}{2}(I_0'^2 - I'^2) - 2C(I_0 - I) \geqq \tfrac{1}{4}\int_I^{I_0}(I'^2 dI/4I), \qquad I < I_0.$$

As $t \to t_0$, $t < t_0$, the integral

$$\int_I^{I_0}(I'^2 dI/I)$$

must be finite, from which it follows that I' must tend to zero.

12. We have

$$(28) \quad 2MT = \int \rho dv \int \rho dv \cdot Sx'^2 = S(\int \rho dv \cdot x')^2$$
$$+ \tfrac{1}{2}S\int\int \rho_1\rho_2 dv_1 dv_2 (x'_1 - x'_2)^2.$$

$$S(x_1 - x_2)^2 \cdot S(x'_1 - x'_2)^2 = r_{12}^2 (r'_{12}^2)^2$$
$$+ S[(x_1 - x_2)(y'_1 - y'_2) - (y_1 - y_2)(x'_1 - x'_2)]^2.$$

$$S(x'_1 - x'_2)^2 = (r'_{12})^2$$
$$+ (1/r_{12}^2)S[(x_1 - x_2)(y'_1 - y'_2) - (y_1 - y_2)(x'_1 - x'_2)]^2,$$

$$(29) \quad 2MT = \tfrac{1}{2}S\int\int \rho_1 dv_1 \rho_2 dv_2 \frac{[(x_1 - x_2)(y'_1 - y'_2) - (y_1 - y_2)(x'_1 - x'_2)]^2}{r_{12}^2}$$
$$+ \tfrac{1}{2}\int\int \rho_1\rho_2 dv_1 dv_2 (r'_{12})^2.$$

Further

$$MI = \tfrac{1}{2}\int\int \rho_1\rho_2 dv_1 dv_2 \cdot r_{12}^2,$$

$$MI' = \int\int \rho_1\rho_2 dv_1 dv_2 r_{12} \cdot r'_{12},$$

$$\int\int \rho_1\rho_2 dv_1 dv_2 (r'_{12})^2 \cdot \int\int \rho_1\rho_2 dv_1 dv_2 r_{12}^2 = (\int\int \rho_1\rho_2 dv_1 dv_2 r_{12} r'_{12})^2$$
$$+ \tfrac{1}{2}\int\int\int\int \rho_1\rho_2\rho_3\rho_4 dv_1 dv_2 dv_3 dv_4 (r_{12}r'_{34} - r_{34}r'_{12})^2,$$

$$\int\int \rho_1\rho_2 dv_1 dv_2 (r'_{12})^2 = M^2 I'^2/2MI$$
$$+ (1/4MI)\int\int\int\int \rho_1\rho_2\rho_3\rho_4 dv_1 dv_2 dv_3 dv_4 (r_{12}r'_{34} - r_{34}r'_{12})^2,$$

$$(30) \quad 2T = 2W + 2C = I'^2/4I + (1/2M)S \int\int \rho_1\rho_2 dv_1 dv_2$$

$$. \frac{[(x_1 - x_2)(y'_1 - y'_2) - (y_1 - y_2)(x'_1 - x'_2)]^2}{r_{12}^2}$$

$$+ (1/8M^2I) \int\int\int\int \rho_1\rho_2\rho_3\rho_4 dv_1 dv_2 dv_3 dv_4 (r_{12}r'_{34} - r_{34}r'_{12})^2.$$

13. Let us denote by S the expression

$$(31) \qquad\qquad S = \int\int \rho_1 dv_1 \rho_2 dv_2 d_{12}.$$

We have

$$SW = \tfrac{1}{2} \int\int \rho_1\rho_2 dv_1 dv_2 \cdot d_{12} \int\int \rho_3\rho_4 dv_3 dv_4 \cdot 1/d_{34}$$

$$= \tfrac{1}{2}\{ (\int\int \rho_1\rho_2 dv_1 dv_2)^2 + \tfrac{1}{2} \int\int\int\int \rho_1\rho_2\rho_3\rho_4 \, dv_1 \cdot dv_2 \cdot dv_3 \cdot dv_4$$

$$\cdot (\sqrt{d_{12}/d_{34}} - \sqrt{d_{34}/d_{12}})^2 \},$$

$$(32) \qquad\qquad SW \geqq \tfrac{1}{2}M^4.$$

We have

$$MI = \tfrac{1}{2} \int\int \rho_1\rho_2 dv_1 dv_2 d_{12},$$

$$M^3I = \tfrac{1}{2} \int\int \rho_3\rho_3 dv_3 dv_4 \cdot \int\int \rho_1\rho_2 dv_1 dv_2 \cdot d_{12}^2 = \tfrac{1}{2}\{ (\int\int \rho_1\rho_2 dv_1 dv_2 \cdot d_{12})^2$$

$$+ \tfrac{1}{2} \int\int\int\int \rho_1\rho_2\rho_3\rho_4 dv_1 dv_2 dv_3 dv_4 \cdot (d_{12} - d_{34})^2 \},$$

$$M^3I \geqq \tfrac{1}{2}S^2.$$

Thus we obtain the inequality

$$(33) \qquad\qquad I \geqq \tfrac{1}{8}(M^5/W^2),$$

from which it follows that if $I \to 0$, $t \to t_1$, we have $W \to + \infty$, $I'' \to + \infty$, I' increases being negative, and the limit of $K^2 \int_{I_0}^{I} T dI$ is finite, from which we conclude that $K = 0$.

It follows from (22) and (33) that

$$(34) \qquad\qquad I'' \geqq 4C + \sqrt{M^5/2I}.$$

14. Let us now study the case $C > 0$. We have

$$I'' > 4C,$$

I'' is always positive, I' increases constantly. Excluding the case $I \to 0$, $t \to t'_1$ we have a single minimum of I and I grows to $+ \infty$ if $t \to + \infty$.

$$I' - I'_0 > 4C(t - t_0).$$

Let us suppose the minimum of I, I_0, realized for $t = t_0$, $I'_0 = 0$. For $t \geqq t_0$ we have

$$dI'/dt = (dI'/dI)I' = \tfrac{1}{2}(d/dI)(I'^2) \geqq 4C + \sqrt{M^5/2I},$$

(35) $\tfrac{1}{2}I'^2 \geqq 4C(I - I_0) + \sqrt{M^5/2} \displaystyle\int_{I_0}^{I} (dI/\sqrt{I}),$

$\tfrac{1}{2}I'^2 \geqq 4C(I - I_0) + \sqrt{2M^5} (\sqrt{I} - \sqrt{I_0}),$

$I'^2 \geqq 8C(I - I_0) + 2\sqrt{2M^5} (\sqrt{I} - \sqrt{I_0}),$

(36) $I' \geqq [8C(I - I_0) + 2\sqrt{2M^5} (\sqrt{I} - \sqrt{I_0})]^{\frac{1}{2}},$

(37) $\displaystyle\int_{I_0}^{I} \{dI/[8C(I - I_0) + 2\sqrt{2M^5} (\sqrt{I} - \sqrt{I_0})]^{\frac{1}{2}}\} \geqq t - t_0.$

Putting $\sqrt{I} = x$, $dI/2\sqrt{I} = dx$, $x_0 = \sqrt{I_0}$ we have

(38) $\displaystyle\int_{x_0}^{x} \{2x dx/[8C(x^2 - I_0) + 2\sqrt{2M^5} (x - \sqrt{I_0})]^{\frac{1}{2}}\} \geqq t - t_0$

$8C(x^2 - I_0) + 2\sqrt{2M^5} (x - \sqrt{I_0})$

$\quad = 8Cx^2 + 2\sqrt{2M^5} x - (8CI_0 + 2\sqrt{2M^5} \cdot \sqrt{I_0})$

$\quad = (\sqrt{8C} x + \sqrt{M^5/2\sqrt{C}})^2 - (8CI_0 + 2\sqrt{2M^5} \sqrt{I_0} + M^5/4C)$

$\quad = (\sqrt{8C} x + \sqrt{M^5/2\sqrt{C}})^2 - (\sqrt{8C} x_0 + \sqrt{M^5/2\sqrt{C}})^2.$

Putting

$$\sqrt{8C} x + \sqrt{M^5/2\sqrt{C}} = s, \qquad \sqrt{8C} x_0 + \sqrt{M^5/2\sqrt{C}} = s_0,$$

we have

(39) $\displaystyle\int_{s_0}^{s} \frac{2[s - \sqrt{M^5/2\sqrt{C}}] \cdot 1/8\sqrt{C} \cdot ds/\sqrt{8C}}{(s^2 - s_0^2)^{\frac{1}{2}}} \geqq t - t_0.$

Neglecting the 2nd term in (34) on the right we get simply

$$\int_{I_0}^{I} dI/\sqrt{8C} \cdot \sqrt{I - I_0} \geqq t - t_0,$$

$$2\sqrt{I - I_0} \cdot 1/\sqrt{8C} \geqq t - t_0,$$

(40) $$l - I_0 \geqq 2C(t - t_0)^2.$$

15. Let us consider the case $C = 0$.

$$I'' \geqq \sqrt{M^5/2I}, \qquad I' - I_0 \geqq \int_{I_0}^{I} \sqrt{M^5/2I}\ dt.$$

I' increases and I has a positive minimum, the case of general collision being excluded, otherwise I' would be always > 0, I would diminish with $t \to -\infty$ and this leads to a contradiction as

$$I' - I' = \int_{t}^{t_0} I'' dt \geqq \int_{t}^{t_0} \sqrt{M^5/2I} \cdot dt,$$

$I' - I'_0 \to -\infty$, with $t \to -\infty$. For $I > I_0$ we have

$$I'^2 \geqq 2\sqrt{2M^5} (\sqrt{I} - \sqrt{I_0}), \qquad I' \geqq [2\sqrt{2M^5} (\sqrt{I} - \sqrt{I_0})]^{\frac{1}{2}}$$

(41) $$\int_{I_0}^{I} \{dI/[2\sqrt{2M^5} (\sqrt{I} - \sqrt{I_0})]^{\frac{1}{2}}\} \geqq t - t_0.$$

Putting $\sqrt{I} = x$, $\sqrt{I_0} = x_0$, we have

$$\int_{x_0}^{x} \{2x dx/[2\sqrt{2M^5} (x - x_0)]^{\frac{1}{2}}\} \geqq t - t_0,$$

or, on setting $\sqrt{x - x_0} = y$, $x = x_0 + y^2$,

(42) $$(2/M)^{5/4}\{x_0 y + y^3/3\} \geqq t - t_0.$$

If $t \to +\infty$, $y \to +\infty$, we have

$$y^3/3 \geqq (\sqrt[4]{M/2})^5 (t - t_0)\{1 + \epsilon(t - t_0)\},$$

$$y \geqq \sqrt[3]{t - t_0} \cdot \sqrt[3]{3} (\sqrt[4]{M/2})^{5/3} \sqrt[3]{t - t_0} \{1 + \epsilon(t - t_0)\},$$

$$(43) \quad I \geqq \{ \sqrt{I_0} + \sqrt[3]{t - t_0}^{\,2} \cdot 3^{2/3} \cdot (M/2)^{5/6} (1 + \epsilon(t - t_0)) \}^2$$
$$= 3^{4/3} (M/2)^{5/3} (t - t_0)^{4/3} \{ 1 + \epsilon(t - t_0) \}.$$

16. We have from (5)

$$u - u(t_0) = \int_{t_0}^{t} \int_{V'} \rho' dv' [\partial(1/d)/\partial x] d\tau,$$

$$v - v(t_0) = \int_{t_0}^{t} \int_{V'} \rho' dv' [\partial(1/d)/\partial y)] d\tau,$$

$$w - w(t_0) = \int_{t_0}^{t} \int_{V'} \rho' dv' [\partial(1/d)/\partial z] d\tau,$$

$$(44) \quad x - x(t_0) - u(t_0)(t - t_0) = \int_{t_0}^{t} d\tau \int_{t_0}^{t} d\tau_1 \int_{V'} \rho' dv' [\partial(1/d)/\partial x],$$

$$y - y(t_0) - v(t_0)(t - t_0) = \int_{t_0}^{t} d\tau \int_{t_0}^{t} d\tau_1 \rho' dv' [\partial(1/d)/\partial y],$$

$$z - z(t_0) - w(t_0)(t - t_0) = \int_{t_0}^{t} d\tau \int_{t_0}^{t} d\tau_1 \rho' dv' [\partial(1/d)/\partial z].$$

Suppose that R is the radius of the least sphere, with center at the center of gravity, which contains all the points of the cosmic cloud. R varies continuously with t. We have the inequalities

$$\mid \int_{V'} \rho' dv' [\partial(1/d)/\partial x] \mid \leqq \int_{V'} \rho' dv' (1/d^2) \text{ etc.,}$$

$$\int_{V'} \rho' dv' \cdot 1/d^2 < \int_{\Sigma_{2R}} \rho' dv' \cdot 1/OP'^2 < \bar{\rho} \int_0^{2R} (4\pi r^2/r^2) dr = \bar{\rho} \cdot 8\pi R,$$

if $\bar{\rho}$ is the upper limit of ρ.

If we suppose that R tends to infinity if $t \to t_1$ (finite), and that the density ρ has the finite upper limit $\bar{\rho}$ we obtain from (44) the inequalities

$$(45) \quad \mid x \mid < \mid x_0 \mid + \mid u_0 \mid (t - t_0) + \int_{t_0}^{t} d\tau \int_{t_0}^{\tau} d\tau_1 \cdot \bar{\rho} 8\pi R(\tau_1),$$

$$\mid y \mid < \mid y_0 \mid + \mid v_0 \mid (t - t_0) + \int_{t_0}^{t} d\tau \int_{t_0}^{\tau} d\tau_1 \bar{\rho} \cdot 8\pi R(\tau_1),$$

$$\mid z \mid < \mid z_0 \mid + \mid w_0 \mid (t - t_0) + \int_{t_0}^{t} d\tau \int_{t_0}^{\tau} d\tau_1 \bar{\rho} \cdot 8\pi R(\tau_1),$$

which lead immediately to a contradiction, if $t_1 - t_0$ is taken sufficiently small.

Thus we obtain

THEOREM 2. *It is not possible for the cosmic cloud to tend to infinity if t tends to a finite value t_1 the density being uniformly limited in t,* $t_0 \leqq t \leqq t_1$.

Indeed (45) holds for all particles within the cloud, but there are particles for which

$$x^2 + y^2 + z^2 = R^2$$

in each moment of time, although it may be that these particles are not the same.

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AN EXPRESSION FOR THE SOLUTION OF A CLASS OF NON-LINEAR INTEGRAL EQUATIONS.*

By R. H. CAMERON and W. T. MARTIN.

1. Introduction. Consider an integral equation

$$(1.1) \qquad x(t) = y(t) + \int_0^t G(t, \xi, x(\xi)) d\xi$$

where $G(t, \xi, u)$ is continuous in

$$(1.2) \qquad 0 \leqq t \leqq 1, \qquad 0 \leqq \xi \leqq 1, \qquad -\infty < u < \infty$$

and satisfies a uniform Lipschitz condition

$$(1.3) \qquad |G(t, \xi, u_2) - G(t, \xi, u_1)| < M |u_2 - u_1|$$

in (1.2). By the usual method of successive approximations it is easily seen that to each continuous function $y(t)$ vanishing at $t = 0$ there corresponds a unique continuous solution $x(t)$. Our purpose in the present paper is not to prove the existence or uniqueness of the solution but rather to give an expression for it. The solution is obtained by taking weighted averages of all continuous functions, with heavier weights on those functions which lie near the solution, that is with heavier weights on those functions $x(t)$ for which the expression

$$(1.4) \qquad \int_0^1 \{y(t) - x(t) + \int_0^t G(t, \xi, x(t)) d\xi\}^2 dt$$

is relatively small. This averaging process is carried out by an integration over the space C of all continuous functions $x(t)$ vanishing at $t = 0$.

For this integration process we could use any integral over C having certain abstract properties. Rather than merely cataloguing the properties which would have to be required of such a general integral we have decided to write this paper in terms of the Wiener integral. This integral seems to be the most satisfactory since it is sufficiently abstract and general and at the same time sufficiently specific to handle the present problem. For the convenience of the reader we shall give in Section 2 a brief resumé of the Wiener integral, indicating the mapping of functions into points and the consequent connection between the Wiener and the Lebesgue integrals.[1]

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[1] A complete treatment of the Wiener integral may be found in the paper, N. Wiener, "Generalized harmonic analysis," *Acta Mathematica*, vol. 55 (1930), pp. 117-258, esp. pp. 214-224.

Our theorem is as follows.

THEOREM 1. *Let $G(t, \xi, u)$ be continuous in* (1.2) *and let it satisfy there the uniform Lipschitz condition* (1.3). *Then if $y(t)$ is any* [2] *continuous function in $0 \le t \le 1$ vanishing at $t = 0$, the integral equation* (1.1) *has a unique continuous solution $x_0(t)$ given by*

$$(1.5) \quad x_0(\tau) = \underset{\rho \to \infty}{\text{l. i. m.}} \frac{\int_C^w \exp[-\rho \int_0^1 \{y(t) - x(t) + \int_0^t G(t, \xi, x(\xi)) d\xi\}^2 dt] x(\tau) d_w x}{\int_C^w \exp[-\rho \int_0^1 \{y(t) - x(t) + \int_0^t G(t, \xi, x(\xi)) d\xi\}^2 dt] d_w x}$$

where the l. i. m. *is taken in the L_2-sense, for ordinary Lebesgue integrals $0 \le \tau \le 1$, and the two integrals $\int_C^w$ are integrals (averages) in the Wiener sense, taken over the space C of all continuous functions $x(\cdot)$ vanishing at the origin.*

A simple example in ordinary algebraic equations will serve to illustrate the general idea of the theorem. Consider, for example, the equation $x^3 = c$; c real. Then the (real) solution x_0 is given by

$$x_0 = \lim_{\rho \to \infty} \frac{\int_{-\infty}^{\infty} \exp[-\rho(x^3 - c)^2] x \, dx}{\int_{-\infty}^{\infty} \exp[-\rho(x^3 - c)^2] dx} \; .$$

For fixed ρ the weighting factor $\exp\{-\rho(x^3 - c)^2\}$ is near zero except for those values of x whose cubes lie near c, while for x a number whose cube lies near c, the weighting factor lies near unity. As ρ becomes larger and larger this process emphasizes more and more those values of x whose cubes lie near c and as $\rho \to \infty$ it yields the solution x_0 whose cube is c.

As we have stated, we shall give a brief resumé of the Wiener integral in the next section. This section may be omitted by those who are familiar with the Wiener integral and by those who are willing to assume that it has certain of the properties of the Lebesgue integral. In Sections **3** and the succeeding sections we introduce and prove a slightly more general theorem and in Section **7** we show that this theorem includes Theorem 1.

2. The Wiener integral.[3] In defining his integral Wiener maps the set

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[2] We state explicitly that this theorem holds for *all* $y(t)$ in C, not merely for almost all. We call attention to this fact because we are using certain concepts which are often associated with the ideas of probability.

[3] See footnote 1.

of all real functions defined on $0 \leqq t \leqq 1$ and vanishing at $t = 0$ into the points on a segment of a line AB of unit length. He makes certain sets of functions $x(t)$ which he calls quasi-intervals, correspond to certain intervals of AB. The quasi-intervals are sets of all functions $x(t)$ defined for $0 \leqq t \leqq 1$ for which

$$(2.1) \quad x(0) = 0; \; x_{j1} \leqq x(t_j) \leqq x_{j2}, \quad [j = 1, \cdots, n; (0 < t_1 < t_2 < \cdots < t_n \leqq 1)].$$

By his definition of measure (which in his terminology would be called probability) the measure of the set of functions $x(t)$ which lie in the quasi-interval is

$$(2.2) \quad \frac{\int_{x_{11}}^{x_{12}} d\xi_1 \cdots \int_{x_{n1}}^{x_{n2}} d\xi_n \exp\{-\xi_1^2/t_1 - \sum_{k=2}^{n} [(\xi_k - \xi_{k-1})^2/(t_k - t_{k-1})]\}}{\pi^{n/2} \sqrt{t_1 (t_2 - t_1)(t_3 - t_2) \cdots (t_n - t_{n-1})}}.$$

In Wiener's theory of random functions this represents the probability that a random function lie in the quasi-interval (2.1). Throughout we shall use the term measure rather than the term probability. Thus the measure of the quasi-interval (2.1) is given by (2.2). As Wiener has pointed out, if the class of all functions $x(t)$ be divided into a finite number of quasi-intervals—some of which then must contain infinite values of x_{j1} or x_{j2}—the sum of their measures will be unity.

Wiener sets up his mapping by a process involving limits of sequences of the quasi-intervals in such a way that the Wiener measure of a quasi-interval is equal to the length of the corresponding interval on AB. Except for a set of points of measure zero, he determines a unique mapping of the points of AB by functions $x(t)$ vanishing at the origin and satisfying

$$(2.3) \qquad\qquad | x(t') - x(t'') | < h \, | \, t' - t'' \, |^{1/4}$$

for some h. Thus a functional of functions $x(t)$ determines a function on the line AB. We note that a functional is Wiener measurable if the corresponding function on the line is Lebesgue measurable. If this function is Lebesgue summable then the Wiener integral of the functional is defined to be the Lebesgue integral of the corresponding function on AB.

Consider a functional $\Phi[x(t_1), x(t_2), \cdots, x(t_n)]$ where $\Phi(\xi_1, \cdots, \xi_n)$ is an ordinary function of the numerical variables $(\xi_1, \cdots, \xi_n)$ for $t_1, t_2, \cdots, t_n$ fixed. If Φ is (Wiener) summable and if $t_1 < t_2 < \cdots < t_n$, then the Wiener integral of Φ is

$$(2.4) \qquad \int_C^w \Phi[x(t_1), \cdots, x(t_n)] d_w x$$

$$= \int_{-\infty}^{\infty} d\xi_1 \cdots \int_{-\infty}^{\infty} d\xi_n \Phi(\xi_1, \cdots, \xi_n)$$

$$\times \frac{\exp\{\xi_1^2/t_1 - (\xi_2 - \xi_1)^2/(t_2 - t_1) - (\xi_n - \xi_{n-1})^2/(t_n - t_{n-1})\}}{\pi^{n/2} \sqrt{t_1 (t_2 - t_1)(t_3 - t_2) \cdots (t_n - t_{n-1})}}.$$

The notation used here differs from that used by Wiener. He writes merely

Average $\{\Phi[x(t_1), \cdots, x(t_n)]\}$ or $\int_0^1 \Phi[x(t_1, \alpha), \cdots, x(t_n, \alpha)] d\alpha$ for the

left member of (2.4). We find it useful to have the above notation. The w above the integral sign in (2.4) indicates that it is a Wiener integral, and the $d_w x$ indicates that the integral is taken with respect to the functions $x(t)$. The C below the integral sign denotes that the integral is taken over all functions $x(t)$ belonging to C, which by (2.3) includes all $x(t)$ except for

a set of measure zero, so that $\int_C^w d_w x = 1$. If $\Phi[x(\cdot)|t]$ is any summable

functional over a measurable subset S of C, then we understand by

$$(2.5) \qquad\qquad\qquad \int_S^w \Phi[x(\cdot)|t] d_w x$$

the integral

$$(2.6) \qquad\qquad\qquad \int_C^w \Psi[x(\cdot)|t] d_w x$$

where

$$(2.7) \qquad\qquad \Psi[x(\cdot)|t] = \begin{cases} \Phi[x(\cdot)|t] & \text{for } x(\cdot) \text{ in } S \\ 0 & \text{otherwise.} \end{cases}$$

A final property which we shall need is expressed in the following lemma.

LEMMA 2.1. *For each $x_0(t) \in C$ and for each $\eta > 0$ the set T_η consisting of all functions $x(t) \in C$ and satisfying*

$$(2.8) \qquad\qquad\qquad \int_0^1 \{x(t) - x_0(t)\}^2 dt < \eta$$

has positive Wiener measure: [4]

$$(2.9) \qquad\qquad\qquad \int_{T_\eta}^w d_w x > 0.$$

---

[4] A consideration of the Wiener mapping, together with the basic equi-continuity property (2.3), will show that $x(\tau)$ is Wiener-Lebesgue measurable as a function of the two variables $x(\cdot)$ and τ in the product space $[x(\cdot) \in C, 0 \leq \tau \leq 1]$. As a con-

sequence the integral $\int_0^1 \{x(t) - x_0(t)\}^2 dt$, for fixed $x_0(t) \in C$, is a Wiener measurable

functional of $x(\cdot)$, and hence the set T_η is Wiener measurable.

This particular property is not encountered, at least not explicitly, in Wiener's paper, it being unnecessary for any portion of that paper. In order to avoid too great a digression at this stage we postpone the proof of the lemma until the final section (Section **8**) and proceed at once to our general theorem, assuming the validity of the lemma for the time being.

3. The general theorem. It seems useful to be able to replace the function $e^{-\lambda\rho}$ which enters into the weighting factor of Theorem 1 by a more general function of two real variables. For this purpose we introduce a function $E(\lambda, \rho)$ which may be any function satisfying the following four conditions.[5]

A. $E(\lambda, \rho) > 0$ for $0 \leqq \lambda < \infty$, $- \infty < \rho < \infty$.

B. For each fixed ρ, $E(\lambda, \rho)$ is continuous in λ in $0 \leqq \lambda < \infty$. For $\rho > 0$, $E(\lambda, \rho)$ is non-increasing in λ and for $\rho < 0$, $E(\lambda, \rho)$ is non-decreasing in λ.

C. $\lim\limits_{\rho \to \infty} \dfrac{E(\lambda, \rho)}{E(\lambda', \rho)} = 0$ for $0 \leqq \lambda' < \lambda$.

D. Corresponding to any three numbers δ, μ, A with $\delta > 0, \mu \geqq 0, A > 0$, there exists a positive number $\rho_0(\delta, A, \mu)$ such that the inequality
$$E(\lambda, \rho)E[(\sqrt{\lambda} + \mu)^2, - A] \leqq E(\delta, \rho)E[(\sqrt{\delta} + \mu)^2, - A]$$
holds for all $\lambda \geqq \delta$ and all $\rho > \rho_0(\delta, A, \mu)$.

We also introduce a general operator $F[x(\cdot)|t]$ defined over the space C of all real functions $\{x(t)\}$ defined and continuous in $0 \leqq t \leqq 1$ and vanishing at $t = 0$. For the special case of Theorem 1 the operator F is simply
$$F[x(\cdot)|t] = x(t) - \int_0^t G(t, \xi, x(\xi))\, d\xi.$$

In our general case the operator F is to take elements $x(t)$ of its domain C into functions $y(t) = F[x(\cdot)|t]$ belonging to C. We assume that the operator has the following four properties.

$1°$. F is continuous in the sense that corresponding to any function

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[5] If $E(\lambda, \rho)$ is taken to be $e^{-\lambda\rho}$ then conditions A, B, and C obviously hold. Condition D can be shown also to hold, with $\rho_0(\delta, \mu, A) = A(1 + \mu\delta^{-\frac{1}{2}})$. To see this we proceed as follows. First $(d/d\xi)[-\rho\xi^2 + A(\xi + \mu)^2] = 2(A - \rho)\xi + 2A\mu$ and this is non-positive whenever $\xi \geqq A\mu/(\rho - A)$ and $\rho > A$. Hence the expression $-\lambda\rho + A(\sqrt{\lambda} + \mu)^2$ is $\leqq -\delta\rho + A(\sqrt{\delta} + \mu)^2$ for $\rho \geqq A(1 + \mu\delta^{-\frac{1}{2}})$. This gives the desired inequality for the exponential.

$\dot{x}_0(t) \, \epsilon \, C$ and to each positive number δ there is a positive number η, depending upon δ and x_0, such that

(3.1) $$\int_0^1 \{F[x(\cdot)\,|\,t] - F[x_0(\cdot)\,|\,t]\}^2 dt \leqq \delta$$

holds whenever

(3.2) $$\int_0^1 \{x(t) - x_0(t)\}^2 dt < \eta.$$

$2°$. F is a $1-1$ transformation of the whole of C into the whole of C. We shall denote the (unique) inverse of F by F^{-1}.

$3°$. F^{-1} is continuous in the sense defined in $1°$ for F.

$4°$. There exist positive constants K and A such that [6]

$$\sqrt{\int_0^1 \{F^{-1}[x(\cdot)\,|\,t]\}^2 dt} \leqq KE\left[\int_0^1 \{x(t)\}^2 dt, -A \right]$$

holds for all $x(t) \, \epsilon \, C$.

We now consider the functional equation

(3.3) $$F[x(\cdot)\,|\,t] = y(t),$$

where $y(t)$ is an arbitrary given function of C. By property $2°$ the solution is unique, being denoted by $F^{-1}[y(\cdot)\,|\,t]$. The theorem which we prove is as follows.

THEOREM 1a. *Let F be an operator which is Wiener Lebesgue measurable in $x(\cdot)$ and t and let it have the properties $1°, \cdots, 4°$. Let $E(\lambda, \rho)$ be a function having the properties $A, \cdots, D$. Then for any $y(t) \, \epsilon \, C$ the (unique) solution of (3.3) is given by*

(3.4) $$F^{-1}[y(\cdot)\,|\,\tau] = \underset{\rho \to \infty}{\text{l. i. m.}} \ \frac{\int_C^w E\left[\int_0^1 \{F[x(\cdot)\,|\,t] - y(t)\}^2 dt, \rho\right] x(\tau) d_w x}{\int_C^w E\left[\int_0^1 \{F[x(\cdot)\,|\,t] - y(t)\}^2 dt, \rho\right] d_w x},$$

where the l. i. m. is taken in the L_2-sense, for ordinary Lebesgue integrals, $0 \leqq \tau \leqq 1$.

Proof. Denote by $x_0(\tau)$ the inverse

(3.5) $$x_0(\tau) = F^{-1}[y(\cdot)\,|\,\tau]$$

---

[6] It should be noted that this is a very weak condition because the second variable of the E-function is negative. Thus if $E(\lambda, \rho)$ were taken to be $e^{-\lambda \rho}$, the right-hand member of the relation in $4°$ would be $K \exp[A \int_0^1 \{x(t)\}^2 dt]$.

which exists and belongs to C by $2°$. Let $\epsilon > 0$ be given. Then by $3°$ there exists a $\delta = \delta(\epsilon, x_0(\cdot))$ such that

$$(3.6) \qquad \int_0^1 \{x(\tau) - x_0(\tau)\}^2 d\tau < \epsilon$$

whenever

$$(3.7) \qquad \int_0^1 \{F[x(\cdot)|\tau] - F[x_0(\cdot)|\tau]\}^2 d\tau < \delta.$$

Notation. We denote by S_δ the set of all $x(\cdot) \, \epsilon \, C$ such that (3.7) holds, and by $\bar{S}_\delta$ its complement in C.

Integrals over S_δ and also over $S_{\delta/2}$ will occur as denominators as we proceed. Thus it will be useful to see that these sets have positive Wiener measure. By property $1°$ of the operator F, there exists an $\eta > 0$ such that $x(\cdot) \, \epsilon \, S_{\delta/2}$ whenever

$$(3.8) \qquad \int_0^1 \{x(\tau) - x_0(\tau)\}^2 d\tau < \eta.$$

We denote by T_η the set of those $x(\cdot)$ for which (3.8) holds. Then $T_\eta \subset S_{\delta/2} \subset S_\delta$. But by Lemma 2.1, T_η has positive Wiener measure; hence $S_{\delta/2}$ and S_δ have positive Wiener measure.

We now write the limitand of (3.4) in the form

$$(3.9) \qquad \frac{\int_C^w E[\int_0^1 \{F[x(\cdot)|t] - y(t)\}^2 dt, \rho]x(\tau)\, d_w x}{\int_C^w E[\int_0^1 \{F[x(\cdot)|t] - y(t)\}^2 dt, \rho]d_w x} = \frac{P_\rho(\tau) + Q_\rho(\tau)}{1 + R_\rho},$$

where

$$(3.10) \qquad P_\rho(\tau) = (1/D_\rho) \int_{S_\delta}^w \Delta_\rho(x, y)x(\tau)\, d_w x$$

$$(3.11) \qquad Q_\rho(\tau) = (1/D_\rho) \int_{\bar{S}_\delta}^w \Delta_\rho(x, y)x(\tau)\, d_w x$$

$$(3.12) \qquad R_\rho = (1/D_\rho) \int_{\bar{S}_\delta}^w \Delta_\rho(x, y)\, d_w x$$

$$(3.13) \qquad D_\rho = \int_{S_\delta}^w \Delta_\rho(x, y)\, d_w x$$

$$(3.14) \qquad \Delta_\rho(x, y) = E[\int_0^1 \{F[x(\cdot)|t] - y(t)\}^2 dt, \rho].$$

It is easily seen that the denominator D_ρ is positive. First, we have already seen that the Wiener measure of S_δ is positive. Next, the integrand is bounded away from zero in $x(\cdot)$ for each fixed positive ρ. This follows

from property B for the function $E(\lambda, \rho)$, together with the fact that (3.7) holds over S_δ. Thus D_ρ is positive for each fixed positive ρ.

Our proof of Theorem 1a will consist of three main steps.

STEP 1.   We shall show that

$$(3.15) \qquad \int_0^1 \{P_\rho(\tau) - x_0(\tau)\}^2 d\tau \leqq \epsilon$$

for all $\rho > 0$.

STEP 2.   We shall show that

$$(3.16) \qquad \lim_{\rho \to \infty} \int_0^1 \{Q_\rho(\tau)\}^2 d\tau = 0.$$

STEP 3.   We shall show that

$$(3.17) \qquad \lim_{\rho \to \infty} R_\rho = 0.$$

In view of the decomposition (3.9) of the limitand the carrying through of these three steps will yield the theorem.

4.   Step 1.   We have

$$
\begin{aligned}
(4.1) \qquad \int_0^1 \{P_\rho(\tau) - x_0(\tau)\}^2 d\tau &= (1/D_\rho^2) \int_0^1 \{D_\rho P_\rho(\tau) -- D_\rho x_0(\tau)\}^2 d\tau \\
&= (1/D_\rho^2) \int_0^1 \{ \int_{S_\delta}^w \Delta_\rho(x, y)[x(\tau) - x_0(\tau)] d_w x \}^2 d\tau \\
&= (1/D_\rho^2) \int_0^1 \{ \int_{S_\delta}^w \Delta_\rho(x^{(1)}, y)[x^{(1)}(\tau) - x_0(\tau)] d_w x^{(1)} \\
&\qquad \cdot \int_{S_\delta}^w \Delta_\rho(x^{(2)}, y)[x^{(2)}(\tau) - x_0(\tau)] d_w x^{(2)} \} d\tau \\
&= (1/D_\rho^2) \int_{S_\delta}^w \int_{S_\delta}^w \{ \Delta_\rho(x^{(1)}, y) \Delta_\rho(x^{(2)}, y) \\
&\qquad \cdot \int_0^1 [x^{(1)}(\tau) - x_0(\tau)][x^{(2)}(\tau) - x_0(\tau)] d\tau \} d_w x^{(1)} d_w x^{(2)} \\
&\leqq (1/D_\rho^2) \int_{S_\delta}^w \int_{S_\delta}^w \{ \Delta_\rho(x^{(1)}, y) \Delta_\rho(x^{(2)}, y) \sqrt{\int_0^1 [x^{(1)}(\tau_1) - x_0(\tau_1)}} \\
&\qquad \cdot \sqrt{\overline{\int_0^1 [x^{(2)}(\tau_2) - x_0(\tau_2)]^2 d\tau_2}} \} d_w x^{(1)} d_w x^{(2)} \\
&= (1/D_\rho^2) \{ \int_{S_\delta}^w \Delta_\rho(x, y) \sqrt{\overline{\int_0^1 [x(\tau) - x_0(\tau)]^2 d\tau}} \, d_w x \}^2.
\end{aligned}
$$

The above operations are justified by the Fubini theorem for Wiener integrals [7] and the Schwarz inequality for Lebesgue integrals.

Over S_δ the inequality (3.6) holds, and hence the Wiener integral in the final member of (4.1) is not greater than $\sqrt{\epsilon}\, \bar{D}_\rho$; and (3.15) follows.

5. Step 2. As in Step 1 it follows from Fubini's theorem and the Schwarz inequality that

$$(5.1) \qquad \int_0^1 \{Q_\rho(\tau)\}^2 d\tau = (1/D_\rho{}^2) \int_0^1 \{ \int_{\bar{S}_\delta}^w \Delta_\rho(x,y) x(\tau) d_w x \}^2 d\tau$$

$$\leq (1/D_\rho{}^2) \{ \int_{\bar{S}_\delta}^w \Delta_\rho(x,y) \sqrt{\int_0^1 [x(\tau)]^2 d\tau}\, d_w x \}^2 .$$

Now by properties 4° and 2° of our operator F, given in **3**, we have

$$(5.2) \qquad \sqrt{\int_0^1 \{x(\tau)\}^2 d\tau} \leq K E[\int_0^1 \{F[x(\cdot)\,|\,t]^2 dt, -A]$$

and by Minkowski's inequality

$$(5.3)\ \int_0^1 \{F[x(\cdot)\,|\,t]\}^2 dt \leq [\sqrt{\int_0^1 \{F[x(\cdot)\,|\,t] - y(t)\}^2 dt} + \sqrt{\int_0^1 \{y(t)\}^2 dt}]^2 .$$

Finally, by property B, Section **3**, the function $E(\lambda, -A)$ is non-decreasing in λ in $0 \leqq \lambda < \infty$. Thus (5.3) and (5.2), when inserted into (5.1), lead to

$$(5.4) \qquad \int_0^1 \{Q_\rho(\tau)\}^2 d\tau \leq (K^2/D_\rho{}^2) \left\{ \int_{\bar{S}_\delta}^w E(\int_0^1 \{F[x(\cdot)\,|\,t] - y(t)\}^2 dt, \rho) \right.$$

$$\left. \cdot E[(\sqrt{\int_0^1 \{F[x(\cdot)\,|\,t] - y(t)\}^2 dt} + \sqrt{\int_0^1 \{y(t)\}^2 dt})^2, -A] d_w x \right\}^2 .$$

Next, we apply property D, Section **3**, of our function $E(\lambda, \rho)$ using the fact that

$$(5.5) \qquad \int_0^1 \{F[x(\cdot)\,|\,t] - y(t)\}^2 dt \geqq \delta$$

holds over $\bar{S}_\delta$. Thus (5.4) yields

$$(5.6) \qquad \sqrt{\int_0^1 \{Q_\rho(\tau)\}^2 d\tau}$$

$$\leq (K/D_\rho) E(\delta, \rho) E[(\sqrt{\delta} + \sqrt{\int_0^1 \{y(t)\}^2 dt}, -A] \int_{\bar{S}_\delta}^w d_w x,$$

---

[7] The Fubini theorem holds for two Wiener integrals or for Wiener and Lebesgue integrals since the Wiener mapping takes function-space into a linear interval to which the ordinary Fubini theorem applies. See Section **2** of the present paper and footnote 1.

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this holding for

(5. 7) $$\rho > \rho_0(\delta, A, \sqrt{\int_0^1 \{y(t)\}^2 dt}).$$

By (3. 13), the definition of $S_{\delta/2}$ (see (3. 7)), and property B of **3** we have

(5. 8) $$D_\rho = \int_{S_\delta}^w \Delta_\rho(x, y) d_u x \geq \int_{S_{\delta/2}}^w \Delta_\rho(x, y) d_w x \geq E(\delta/2, \rho) \int_{S_{\delta/2}}^w d_w x.$$

Inserting this into (5. 6) and using the fact that $\int_{\bar{S}_\delta}^w d_w x \leq \int_C^w d_w x = 1$, we find that

(5. 9) $$\sqrt{\int_0^1 \{Q_\rho(\tau)\}^2 d\tau} \leq K \frac{E(\delta, \rho)}{E(\delta/2, \rho)} \frac{E[(\sqrt{\delta} + \sqrt{\int_0^1 \{y/t)\}^2 dt}, -A]}{\int_{S_{\delta/2}}^w d_w x}$$

holds for all ρ satisfying (5. 7). By property C of **3**

$$\lim_{\rho \to \infty} \frac{E(\delta, \rho)}{E(\delta/2, \rho)} = 0,$$

and thus (3. 16) holds.

    6. Step 3. In view of property B of **3** and the definition (3. 14) of $\Delta_\rho(x, y)$ it follows that

(6. 1) $$\Delta_\rho(x, y) \leqq E(\delta, \rho)$$

holds over $\bar{S}_\delta$ and

(6. 2) $$\Delta_\rho(x, y) \geqq E(\delta, \rho)$$

holds over S_δ. Hence

(6. 3) $$R_\rho = \frac{\int_{\bar{S}_\delta}^w \Delta_\rho(x, y) d_w x}{\int_{S_\delta}^w \Delta_\rho(x, y) d_w x} \leq \frac{E(\delta, \rho) \int_C^w d_w x}{\int_{S_{\delta/2}}^w \Delta_\rho(x, y) d_w x}$$

$$\leqq \frac{E(\delta, \rho)}{E(\delta/2, \rho)} \frac{1}{\int_{S_{\delta/2}}^w d_w x} .$$

The desired relation (3. 17) follows by property C of **3**, together with the fact that $S_{\delta/2}$ has positive measure.

    This concludes the proof of Theorem 1a. In the next section we shall show that Theorem 1a includes Theorem 1 as a special case, and in the final

section, **8**, we shall give a proof of Lemma 2. 1 which stated that the set S_δ has positive Wiener measure.

7. Theorem 1 as a special case of Theorem 1a. We have already seen that the function $e^{-\lambda\rho}$ satisfies the four conditions A, $\cdots$, D laid down in **3** for the function $E(\lambda, \rho)$. (Cf. footnote 5.) Hence all that remains to prove that Theorem 1 is a special case of Theorem 1a is to show that the special functional

$$(7.1) \qquad F[x(\cdot)\,|\,t] = x(t) - \int_0^t G(t, \xi, x(\xi))\,d\xi$$

is Wiener Lebesgue measurable in $x(\cdot)$ and t and that it satisfies the four conditions $1°, \cdots, 4°$ laid down on F in **3** where, in condition $4°$, $E(\lambda, \rho)$ is to be replaced by $e^{-\lambda\rho}$. We now proceed to show this.

First, since $x(t)$ is Wiener Lebesgue measurable in $x(\cdot)$ and t (see Footnote 4) and since $G(t, \xi, u)$ is continuous in t, ξ, u it follows that the functional (7. 1) is Wiener Lebesgue measurable in $x(\cdot)$ and t.

Next, we look into the continuity of the operator (7. 1). If $x'(t)$ and $x''(t)$ are any two functions belonging to C then by Minkowski's inequality and the Lipschitz condition (1. 3), we have

$$(7.2) \quad [\int_0^1 \{x'(t) - x''(t) - \int_0^t (G(t, \xi, x'(\xi)) - G(t, \xi, x''(\xi)))\,d\xi\}^2 dt]^{1/2}$$

$$\leqq [\int_0^1 \{x'(t) - x''(t)\}^2 dt]^{1/2} + [\int_0^1 \{\int_0^t (G(t, \xi, x'(\xi)) - G(t, \xi, x''(\xi)))\,d\xi\}^2 dt]^{1/2}$$

$$\leqq [\int_0^1 \{x'(t) - x''(t)\}^2 dt]^{1/2} + [\int_0^1 \{\int_0^t M\,|\,x'(\xi) - x''(\xi)|\,d\xi\}^2 dt]^{1/2}$$

$$\leqq (1 + M)[\int_0^1 \{x'(t) - x''(t)\}^2 dt]^{1/2}.$$

This yields the desired property $1°$ for the operator (7. 1) (even in a somewhat stronger form).

Clearly, if $x(t)$ is any function belonging to C, then the left member of (7. 1) defines a unique function $y(t)$ belonging to C. Conversely, if $y(t)$ is any function belonging to C, then by the usual method of successive approximations one shows easily that the integral equation (1. 1) possesses a unique solution $x(t)$ belonging to C. The Lipschitz condition (1. 2) is used freely in this proof. The procedure is so much a standard one that we omit the details. These two facts show that the operator (7. 1) possesses property $2°$ of **3**.

To show that our operator possesses properties $3°$ and $4°$ we first prove the following lemma.

LEMMA 7.1. *If* $y'(t), y''(t)$ *are any two functions of C and* $x'(t), x''(t)$
the corresponding (unique) solutions of (1.1), then

$$(7.3) \qquad \sqrt{\int_0^1 \{x'(t) - x''(t)\}^2 dt} \leq [\sum_{n=0}^{\infty} (M^n/\sqrt{n!})] \sqrt{\int_0^1 \{y'(t) - y''(t)\}^2 dt.}$$

Proof. For each of the functions $y^{(j)}(t), (j = 1, 2)$, we define the approximate functions

$$(7.4) \qquad x_0^{(j)}(t) = y^{(j)}(t)$$

$$x_{n+1}^{(j)}(t) = y^{(j)}(t) + \int_0^t G(t, \xi, x_n^{(j)}(\xi)) d\xi, \qquad (n = 0, 1, 2, \cdots).$$

Then

$$(7.5) \quad |x'_{n+1}(t) - x''_{n+1}(t)|$$

$$= |y'(t) - y''(t) + \int_0^t [G(t, \xi, x_n'(\xi)) - G(t, \xi, x_n''(\xi))] d\xi|$$

$$\leq |y'(t) - y''(t)| + M \int_0^t |x_n'(\xi) - x_n''(\xi)| d\xi$$

$$\leq |y'(t) - y''(t)| + M \sqrt{\int_0^t |x_n'(\xi) - x_n''(\xi)|^2 d\xi};$$

$$(n = 0, 1, 2, \cdots; 0 \leq t \leq 1).$$

We now make an induction assumption, namely for a fixed index $k \geq 0$
we assume that

$$(7.6) \quad \sqrt{\int_0^t \{x_k'(\xi) - x_k''(\xi)\}^2 d\xi} \leq \sqrt{\int_0^t \{y'(\xi) - y''(\xi)\}^2 d\xi} \sum_{n=1}^{k+1} \frac{M^{n-1} t^{(n-1)/2}}{\sqrt{(n-1)!}}$$

$$= \sqrt{\int_0^t \{y'(\xi) - y''(\xi)\}^2 d\xi} \sum_{n=0}^{k} \frac{M^n t^{n/2}}{\sqrt{n!}}$$

holds for all t in $0 \leq t \leq 1$. On inserting (7.6) into (7.5), with $n = k$,
we find

$$(7.7) \quad |x'_{k+1}(t) - x''_{k+1}(t)|$$

$$\leq |y'(t) - y''(t)| + M \sqrt{\int_0^t \{y'(\xi) - y''(\xi)\}^2 d\xi} \sum_{n=0}^{k} \frac{M^n t^{n/2}}{\sqrt{n!}}$$

holds for $0 \leq t \leq 1$. Hence (7.7) and the Minkowski inequality yield

$$(7.8) \quad \sqrt{\int_0^t \{x'_{k+1}(\xi) - x''_{k+1}(\xi)\}^2 d\xi}$$

$$\leqq \sqrt{\int_0^t \{y'(\xi) - y''(\xi)\}^2 d\xi}$$

$$+ \sum_{n=0}^k \frac{M^{n+1}}{\sqrt{n!}} \sqrt{\int_0^t \left[\int_0^\xi \{y'(\zeta) - y''(\zeta)\}^2 d\zeta\right]\xi^n d\xi}$$

$$\leqq \sqrt{\int_0^t \{y'(\xi) - y''(\xi)\}^2 d\xi}\left[1 + \sum_{n=0}^k \frac{M^{n+1}}{\sqrt{n!}}\sqrt{\int_0^t \xi^n d\xi}\right]$$

$$= \sqrt{\int_0^t \{y'(\xi) - y''(\xi)\}^2 d\xi}\left[1 + \sum_{n=0}^k \frac{M^{n+1}}{\sqrt{(n+1)!}} t^{(n+1)/2}\right]$$

$$= \sqrt{\int_0^t \{y'(\xi) - y''(\xi)\}^2 d\xi} \sum_{n=0}^{k+1} \frac{M^n}{\sqrt{n!}} t^{n/2}.$$

Thus, under the assumption that (7.6) holds for the index k, we find that it also holds for the index $k+1$. But for $k=0$, (7.6) does hold since $x_0^{(j)}(t) = y^{(j)}(t)$. Hence our induction process is complete and (7.6) holds for all $k = 0, 1, 2, \cdots$. Now by the classical theory of successive approximations the approximating functions $x_n^{(j)}(t)$ converge (uniformly in $0 \leqq t \leqq 1$) to the solution $x^{(j)}(t)$. Hence on taking the limit in (7.6) as $k \to \infty$ and putting $t = 1$, we obtain the desired inequality (7.3) of the lemma.

This shows that the inverse operator to (7.1) is continuous in the mean-square sense, that is, the operator (7.1) possesses the property 3°.

To show that our operator possesses the property 4° we consider a special case of Lemma 7.1, namely that in which $x''(t)$ is identically zero. We denote the corresponding $y''(t)$ by $y^*(t)$, which is given explicitly by $-\int_0^t G(t, \xi, 0) d\xi$. With $x''(t)$ and $y''(t)$ so taken, Lemma 7.1 yields

$$(7.9) \quad \sqrt{\int_0^1 \{x'(t)\}^2 dt} \leqq \Gamma_M \sqrt{\int_0^1 \{y'(t) - y^*(t)\}^2 dt}$$

$$\leqq \Gamma_M \sqrt{\int_0^1 \{y'(t)\}^2 dt} + \Gamma^*$$

where

$$(7.10) \quad \Gamma_M = \sum_{n=0}^\infty (M^n/\sqrt{n!}), \quad \Gamma^* = \sqrt{\int_0^1 \{y^*(t)\}^2 dt}\, \Gamma_M.$$

This leads us at once to

$$(7.11) \quad \sqrt{\int_0^1 \{x'(t)\}^2 dt} \leqq (1 + \Gamma^*) \exp\left[\Gamma_M^2 \int_0^1 \{y'(t)\}^2 dt\right].$$

Thus if $y'(t)$ is any function of C and $x'(t)$ the corresponding solution of (1.1), then (7.10) holds. Due to property 2° this yields property 4° for an operator 7.1, where in property 4° the function $E(\lambda, \rho)$ is replaced by $e^{-\lambda\rho}$.

Thus the operator 7.1 possesses the four requisite properties and hence Theorem 1 is a special case of Theorem 1a.

8. Proof of Lemma 2.1. In this final section we give a proof of Lemma 2.1. Although the conclusion seems very reasonable we have not been able to construct a very easy proof of it. We now prove the lemma for the case when the given function $x_0(t)$ happens to have a continuous first derivative in $0 \leq t \leq 1$, i. e. we prove

LEMMA 8.1. *Let $x^*(t)$ be any function of C which possesses a continuous first derivative in $0 \leq t \leq 1$, and let η be any positive number. Then*

$$(8.1) \qquad\qquad \int_{T^*_\eta}^w d_w x > 0$$

where T^*_η is set of all functions $x(t)$ of C for which

$$(8.2) \qquad\qquad \int_0^1 \{x(t) - x^*(t)\}^2 dt < \eta.$$

Once we have proved Lemma 8.1 our general result (Lemma 2.1) will be easily obtained as follows. Let $x_0(t)$ be any function of C, and let η be any positive number. Then, by the Weierstrass approximation theorem, there exists a polynomial of C, say $x^*(t)$, such that

$$(8.3) \qquad\qquad \int_0^1 \{x^*(t) - x_0(t)\}^2 dt < \eta/4.$$

And by the Minkowski inequality

$$(8.4) \qquad \sqrt{\int_0^1 \{x(t) - x_0(t)\}^2 dt} \leq \sqrt{\int_0^1 \{x(t) - x^*(t)\}^2 dt}$$
$$+ \sqrt{\int_0^1 \{x^*(t) - x_0(t)\}^2 dt}.$$

Hence

$$(8.5) \qquad\qquad T^*_{\eta/4} \subset T_\eta,$$

and thus if T^*_η has positive measure for every $\eta > 0$, then T_η also does. Hence it is sufficient to prove Lemma 8.1.

For the proof of Lemma 8.1 let n be a positive integer and form the functions

$$(8.6) \quad a_{k,n}(t) = [x^*((k-1)/n) - x^*(t)][k-nt]$$
$$+ [x^*(k/n) - x^*(t)][nt - (k-1)];$$
$$(k = 1, \cdots, n; \; n = 1, 2, \cdots),$$

and

$$(8.7) \qquad A_n = \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} \{a_{k,n}(t)\}^2 dt.$$

Since $x^*(t)$ has a continuous first derivative in $0 \leq t \leq 1$ there exists a positive constant ρ such that

$$(8.8) \qquad \{a_{k,n}(t)\}^2 \leq 4\rho^2 n^2 [(k-1)/n - t]^2 [k/n - t]^2,$$

and hence

$$(8.9) \quad A_n \leq 4\rho^2 n^2 \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} [(k-1)/n - t]^2 [k/n - t]^2 dt$$
$$\leq (4\rho^2/n^2) \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} dt = 4\rho^2/n^2.$$

Thus

$$(8.10) \qquad \lim_{n \to \infty} A_n = 0.$$

We shall need (8.10) later in our proof. We also list four preliminary formulas, all very elementary. The first three are

$$(8.11) \quad (1/\sqrt{\pi}) \int_{-\infty}^{\infty} e^{-\xi^2} d\xi = 1, \; (1/\sqrt{\pi}) \int_{-\infty}^{\infty} \xi e^{-\xi^2} d\xi = 0, \; (1/\sqrt{\pi}) \int_{-\infty}^{\infty} \xi^2 e^{-\xi^2} d\xi = \tfrac{1}{2}.$$

The fourth formula is

$$(8.12) \quad \alpha(1-\alpha)[(\xi-a)^2/\alpha + (\xi-b)^2/(1-\alpha)]$$
$$= \xi^2 - 2\xi[a(1-\alpha) + b\alpha] + a^2(1-\alpha) + b^2\alpha$$
$$= \xi^2 - 2\xi[a(1-\alpha) + b\alpha] + [a(1-\alpha) + b\alpha]^2$$
$$+ a^2\alpha(1-\alpha) + b^2\alpha(1-\alpha) - 2ab\alpha(1-\alpha)$$
$$= \{\xi - [a(1-\alpha) + b\alpha]\}^2 + (a-b)^2\alpha(1-\alpha).$$

We proceed now with the proof of Lemma 8.1. Let η be any positive number and let $x^*(t)$ be any function of C having a continuous derivative in $0 \leq t \leq 1$. Then by (8.10) the quantity A_n of (8.5) has the limit zero. Next let n be a positive integer such that

$$(8.13) \qquad 1/12n + A_n < \eta.$$

In the remainder of the proof n will be a fixed integer satisfying (8.13). For λ a positive number we denote by I_λ the set (Wiener quasi-interval) of all $x(t)$ of C satisfying

(8.14) $-\lambda < x(k/n) - x^*(k/n) < \lambda$, for $(k = 1, \cdots, n)$,

and let

(8.15) $Q_\lambda = [1/(2\lambda)^n] \int_{I_\lambda}^w \{\eta - \int_0^1 [x(t) - x^*(t)]^2 dt\} d_w x$.

For convenience in writing we abbreviate

(8.16) $x^*(k/n) = v_k$, $(k = 1, \cdots, n)$,

(8.17) $c_k = \exp\{-n[v_1{}^2 + (v_2 - v_1)^2 + \cdots$
$\qquad\qquad + (v_{k-1} - v_{k-2})^2 + (v_{k+1} - v_k)^2 + \cdots + (v_n - v_{n-1})^2]\}$,

(8.18) $c = \exp[-n(v_k - v_{k-1})^2]c_k$.

On performing a justifiable interchange of order of integration in (8.15) and using the expression (2.4) for the Wiener integral we obtain

(8.19) $Q_\lambda = [1/(2\lambda)^n] \int_0^1 \{\int^w [\eta - (x(t) - x^*(t))^2] d_w x\} dt$

$\qquad = [1/(2\lambda)^n] \sum_{k=1}^n \int_{(k-1)/n}^{k/n} \{\int_{I_\lambda}^w [\eta - (x(t) - x^*(t))^2] d_w x\} dt$

$\qquad = [1/(2\lambda)^n] \sum_{k=1}^n$

$\int_{(k-1)/n}^{k/n} \{\dfrac{1}{\pi^{(n+1)/2}(1/n)^{(k-1)/2}\sqrt{(t - (k-1)/n)(k/n - t)}\,(1/n)^{(n-k)}}$

$\cdot \int_{-\lambda+v_1}^{\lambda+v_1} d\xi_1 \cdots \int_{-\lambda+v_{k-1}}^{\lambda+v_{k-1}} d\xi_{k-1} \int_{-\infty}^{\infty} d\xi \int_{-\lambda+v_k}^{\lambda+v_k} d\xi_k \cdots \int_{-\lambda+v_n}^{\lambda+v_n} d\xi_n$

$\cdot [\exp n(\xi_1{}^2 - (\xi_2 - \xi_1)^2 - \cdots - (\xi_k - \xi_{k-1})^2$

$\quad - \dfrac{(\xi - \xi_{k-1})^2}{nt - (k-1)} - \dfrac{(\xi_k - \xi)^2}{k - nt} - (\xi_{k+1} - \xi_k)^2 - \cdots - (\xi_n - \xi$

$\cdot [\eta - (\xi - x^*(t))^2]\} dt$

$\qquad = \sum_{k=1}^n (n/\pi)^{(n+1)/2} \int_{(k-1)/n}^{k/n} dt \, \dfrac{1}{\sqrt{[nt - (k-1)][k - nt]}}$

$\cdot \int_{-\infty}^{\infty} d\xi[\eta - (\xi - x^*(t))^2](1/2\lambda) \int_{-\lambda+v_1}^{\lambda+v_1} d\xi_1 \cdots$

$\cdot (1/2\lambda) \int_{-\lambda+v_n}^{\lambda+v_n} d\xi_n \exp\{-n[\xi_1{}^2 + (\xi_2 - \xi_1)^2 + \cdots + (\xi_{k-1} - \xi_{k-2})^2]\}$

$\cdot \exp\{-n[(\xi_{k+1} - \xi_k)^2 + \cdots + (\xi_n - \xi_{n-1})^2]\}$

$\cdot \exp \{-\dfrac{n(\xi - \xi_{k-1})^2}{nt - (k-1)} - \dfrac{n(\xi_k - \xi)^2}{k - nt}\}$.

Now Q_λ contains an n-fold average over an n-dimensional cube of side 2λ, and since the integrand is continuous, the limit of the average as $\lambda \to 0$ is the value of the integrand at the center of the n-cube. Also the limit may be taken equally well before or after the t and ξ integrations since the integrand of the n-fold integral is in absolute value less than

$$\exp\{-n[|\xi| - |v_k| - \lambda]^2\}$$

for sufficiently large $|\xi|$. Hence

$$(8.20) \quad \lim_{\lambda \to 0} Q_\lambda = (n/\pi)^{(n+1)/2} \sum_{k=1}^{n} c_k \int_{(k-1)/n}^{k/n} dt \; \frac{1}{\sqrt{(nt-k+1)(k-nt)}} \; \mathfrak{z}_k(t)$$

where

$$(8.21) \quad \mathfrak{z}_k(t) = \int_{-\infty}^{\infty} d\xi [\eta - (\xi - x^*(t))^2]$$

$$\cdot \exp \left\{ - \frac{n(\xi - v_{k-1})^2}{nt-k+1} - \frac{n(v_k - \xi)^2}{k-nt} \right\}$$

$$= \int_{-\infty}^{\infty} d\xi (\eta - \xi^2) \exp \left\{ - \frac{n(\xi + x^*(t) - v_{k-1})^2}{nt-k+1} - \frac{n(\xi + x^*(t) - v_k)^2}{k-nt} \right\}.$$

Now by (8.12) with $a = v_{k-1} - x^*(t)$, $b = v_k - x^*(t)$, $\alpha = nt - k + 1$, we see that

$$(8.22) \quad \mathfrak{z}_k(t) = \exp[- n(v_k - v_{k-1})^2] \int_{-\infty}^{\infty} d\xi (\eta - \xi^2) \exp \frac{-n[\xi - a_{k,n}(t)]^2}{(nt-k+1)(k-nt)},$$

where the $a_{k,n}(t)$ are defined in (8.6). On making the change of variable

$$(8.23) \quad \sqrt{\frac{n}{(nt-k+1)(k-nt)}} \; (\xi - a_{k,n}(t)) \to \xi$$

and using (8.11) we have

$$(8.24) \quad \mathfrak{z}_k(t) = \sqrt{\frac{(nt-k+1)(k-nt)}{n}} \; \exp[- n(v_k - v_{k-1})^2] \int_{-\infty}^{\infty} d\xi \exp(-\xi^2).$$

$$\cdot \left[\eta - \frac{(nt-k+1)(k-nt)}{n} \xi^2 - a_{k,n}^2(t) \right]$$

$$= \exp[- n(v_k - v_{k-1})^2] \sqrt{\frac{(nt-k+1)(k-nt)}{n}} \; \pi$$

$$\left[\eta - a_{k,n}^2(t) - \tfrac{1}{2} \frac{(nt-k+1)(k-nt)}{n} \right].$$

We now return to (8.20) using (8.7), (8.18) and (8.24) to simplify it.

$$(8.25) \quad \lim_{\lambda \to 0} Q_\lambda = (n/\pi)^{n/2} c \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} \left\{ \eta - a^2_{k,n}(t) - \frac{(nt - k + 1)(k - nt)}{2n} \right.$$

$$= (n/\pi)^{n/2} c \sum_{k=1}^{n} \left[\eta t - \frac{1}{2n^2} \left\{ \frac{(nt - k + 1)^2}{2} - \frac{(nt - k + 1)^3}{3} \right\} \right]_{(k-1)/n}^{k/n} - (n/\pi)^{}$$

$$= (n/\pi)^{n/2} c \sum_{k=1}^{n} \left[\frac{\eta}{n} - \frac{1}{2n^2} \left(\tfrac{1}{2} - \tfrac{1}{3} \right) \right] - (n/\pi)^{n/2} c A_n$$

$$= (n/\pi)^{n/2} c [\eta - 1/12n - A_n].$$

Since n was originally chosen so that (8.13) holds, it follows that

$$(8.26) \qquad\qquad\qquad\qquad \lim_{\lambda \to 0} Q_\lambda > 0.$$

Hence there exists a $\lambda_0 > 0$ such that

$$(8.27) \qquad\qquad\qquad\qquad Q_{\lambda_0} > 0;$$

or

$$(8.28) \qquad\qquad \int_{I_{\lambda_0}}^{w} [\eta - \int_0^1 \{x(t) - x^*(t)\}^2 dt] d_w x > 0.$$

But

$$(8.29) \qquad\qquad I_{\lambda_0} = I_{\lambda_0} \cdot T^*\eta + I_{\lambda_0} \cdot [C - T^*\eta]$$

so that

$$(8.30) \qquad \int_{I_{\lambda_0} \cdot T^*\eta}^{w} [\eta - \int_0^1 \{x(t) - x^*(t)\}^2 dt] d_w x$$

$$> \int_{I_{\lambda_0} \cdot [C - T^*\eta]}^{w} [\int_0^1 \{x(t) - x^*(t)\}^2 dt - \eta] d_w x.$$

Now when $x(t)$ is in $C - T^*\eta$ the inequality

$$(8.31) \qquad\qquad\qquad \int_0^1 \{x(t) - x^*(t)\}^2 dt \geqq \eta$$

holds so that the right member of (8.30) is non-negative, and the left member is positive. Hence the Wiener measure of $T^*\eta$ is positive, for if $T^*\eta$ were a null-set, $I_{\lambda_0} \cdot T^*\eta$ would also be a null set and the integral over it would be zero. This yields Lemma 8.1, and by the remarks made immediately following the statement of Lemma 8.1 this also yields Lemma 2.1.

THE MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

ON THE ABSOLUTE CESÀRO SUMMABILITY OF NEGATIVE ORDER FOR A FOURIER SERIES AT A GIVEN POINT.*

By KIEN-KWONG CHEN.

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1. Introduction. We suppose throughout that $f(t)$ is a periodic function with period 2π, integrable in the Lebesgue sense, and that

$$(1.1) \qquad p > 1, \qquad 0 < k < 1.$$

Fixing x we write $\psi(t)$ for the conjugate of the function $\frac{1}{2}\{f(x+t) + f(x-t)\}$ and set

$$(1.2) \qquad \alpha_0 = \max(\tfrac{1}{2} - k, 1/p - k).$$

The principal theorem in this paper is Theorem 3 which contains the following

THEOREM 1. *Suppose that $pk > 1$; and for a given point x,*

$$(1.3) \qquad \int_0^\pi | \psi(t+h) - \psi(t-h)|^p \, dt = O(h^{pk}), \qquad (h \to +0)$$

then the Fourier series of $f(t)$, at x, is summable $|C, \alpha|$, when $\alpha > \alpha_0$.

By means of Theorem 1, we generalize Zygmund's theorem [1] concerning the absolute convergence for Fourier series as follows.

THEOREM 2. *If $f(t)$ is of bounded variation and belongs to Lip k, then the Fourier series of $f(t)$ is summable $|C, \alpha|$, when $\alpha > -\frac{1}{2}k$; and is summable (C, β), when*

$$\beta > -\tfrac{1}{2}k - \tfrac{1}{2}.$$

It is well-known that if $pk > 1$ and $p \leq 2$, then the relation

$$(1.4) \qquad \int_{-\pi}^{\pi} | f(x+h) - f(x-h)|^p \, dx = O(h^{pk})$$

implies the absolute convergence of the Fourier series of $f(t)$.[2] This result is improved by Theorem 1, since in the present case

$$\alpha_0 = 1/p - k < 0.$$

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[1] Zygmund [10].

[2] Hardy and Littlewood [2], Theorem 8.

Hyslop [3] proves that if $f(t) \epsilon$ Lip k, and $2k \leqq 1$, then the Fourier series is summable $|C, \alpha|$, when $\alpha > \frac{1}{2} - k$. The same conclusion follows from the less stringent condition (1.4), provided that

$$1/p < k \leqq \tfrac{1}{2}.$$

This result is due to Chow,[4] and is evidently included in Theorem 1.

THEOREM 3. *Suppose that for the point x, there is a number q such that*

(1.5) $q + pk > 1$

and that as $h \to +0$, the condition

(1.6) $\int_{-\pi}^{\pi} |\psi(t + h) - \psi(t - h)|^p t^{-q} dt = O(h^{pk})$

holds. Then the Fourier series of $f(t)$, at $t = x$, is summable $|C, \alpha|$, when $\alpha > \alpha_0$, and is summable (C, β), when $\beta > -k$.

The last clause generalizes a known theorem [5] on (C, α) summability of Fourier series.

Our main purpose, then, is to obtain criteria for the Cesàro summability of Fourier series; but we also prove some theorems on power series. We prove, for example, that if the function

(1.7) $F(z) = \sum\limits_{n=0}^{\infty} c_n z^n \qquad (z = re^{i\theta})$

is regular in the unit circle, and the relation

(1.8) $\int_{-\pi}^{\pi} |F^{(j)}(z)|^p d\theta = O((1 - r)^{kp-jp}), \qquad (r \to 1 - 0)$

holds for some positive integer j, then the series (1.7) is summable $|C, \alpha|$, when $\alpha > \alpha_0$, at every point of the unit circle where the function is regular. This result is a corollary of the following

THEOREM 4. *If $F(z) = \Sigma c_n z^n (z = re^{i\theta})$ is regular in the unit circle, and for a positive integer j,*

(1.9) $\int_{-\pi}^{\pi} \dfrac{|F^{(j)}(z)|^p d\theta}{|1 - z|^q} = O((1 - r)^{kp-jp}), \qquad (r \to 1 - 0)$

then the series Σc_n is summable $|C, \alpha|$, whenever $\alpha > \alpha_0$ and $0 \leqq q \leqq 1$, $p > 1$, $0 < k < 1$, $q + pk > 1$.

This theorem, for the special case $j = 1$, $q = 0$ and

[3] Hyslop [7], Theorem 1.
[4] Chow [1], Theorem 3.
[5] Hardy and Littlewood [2], Theorem 7.

$$(1.10) \qquad\qquad 1/p < k \leqq \tfrac{1}{2},$$

is due to Chow.[6] It should be observed that the sufficient condition [7]

$$(1.11) \qquad G(r, t) = \int_0^t | F'(re^{i\theta+i\phi}) |^p d\phi = O\left(\frac{|t|}{(1-r)^{p-pk}}\right)$$

$$(pk \leqq 1,\ 0 < 1-r \leqq |t| \leqq \pi)$$

for the summability $|C, \alpha|$, $\alpha > \alpha_0$, of the series $\Sigma c_n e^{ni\theta} = F(e^{i\theta})$ implies the existence of a number q which satisfies (1.5) and

$$(1.12) \qquad\qquad \int_{-\pi}^{\pi} \frac{| F'(re^{i\theta+i\phi}) |^p d\phi}{| 1 - re^{i\phi} |^q} = O((1-r)^{kp-p}).$$

A derivation of this result is given in **4**.

2. A lemma concerning conjugate functions.

LEMMA 1. *Let $u(\theta)$ be an even function integrable in $(0, \pi)$, and periodic with period 2π; then the conjugate function*

$$(2.1) \qquad\qquad v(\theta) = (1/\pi) \int_0^{\pi} \frac{\sin\theta}{\cos\phi - \cos\theta}\, u(\phi) d\phi$$

satisfies the relation

$$(2.2) \qquad\qquad \int_0^{\pi} | v(\theta) |^p \theta^{-q} d\theta \leqq K(p, q) \int_0^{\pi} | u(\theta) |^p \theta^{-q} d\theta$$

provided that $p > 1$, $-p < q-1 < p$, $\theta^{-q}\, | u(\theta) |^p \, \epsilon\, L(0, \pi)$.

This theorem is substantially known.[8] We give here a proof, for the sake of completeness.

Without loss of generality, we may assume that $u(\theta)$ vanishes in the interval $(\pi - \delta, \pi)$, where

$$\tfrac{1}{2}\pi < \delta < \pi.$$

In fact, let

$$u(\theta) = u_1(\theta) + u_2(\theta),$$
$$u_1(\theta) = u(\theta) \qquad (0 \leqq \theta \leqq \pi - \delta),$$
$$u_1(\theta) = 0 \qquad (\pi - \delta < \theta \leqq \pi),$$

and let the conjugate of $u_j(\theta)$ be denoted by $v_j(\theta)$ which is given by a formula like (2.1). Then

[6] Chow [1], Theorem 1.

[7] Chow [1], Theorem 2.

[8] Hardy and Littlewood [3], Theorem 11.

$$\int_0^\pi \frac{|v_2(\theta)|^p d\theta}{\theta^q} = \int_{\pi/2}^\pi \frac{|v_2(\theta)|^p d\theta}{\theta^q} + \int_0^{\pi/2} \frac{d\theta}{\theta^q} \left| \int_{\pi-\delta}^\pi \frac{\sin\theta u_2(\phi) d\phi}{\cos\phi - \cos\theta} \right|^p$$

$$\leqq \pi^p \int_0^\pi |v_2(\theta)|^p d\theta + (\pi\cos\delta)^{-p} \int_0^{\pi/2} \theta^{p-q} d\theta \int_0^\delta |u(\pi-t)|^p dt.$$

This is less than a constant multiple of $\int |u_2|^p d\theta$, by Riesz's inequality. Hence

$$(2.3) \qquad \int_0^\pi |v_2(\theta)|^p \theta^{-q} d\theta \leqq K(p,q) \int_0^\pi |u_2(\theta)|^p \theta^{-q} d\theta.$$

and (2.2) follows from (2.3) and

$$\left(\int |v|^p \theta^{-q} d\theta \right)^{1/p} \leqq \sum_1^2 \left(\int |v_j|^p \theta^{-q} d\theta \right)^{1/p}$$

provided that

$$\int_0^\pi |v_1(\theta)|^p \theta^{-q} d\theta \leqq K(p,q) \int_0^\pi |u_1(\theta)|^p \theta^{-q} d\theta.$$

Let

$$V(\theta) = (1/\pi) \int_0^\pi \frac{\sin\theta}{\cos\phi - \cos\theta} U(\phi) d\phi$$

be the conjugate of the even function

$$U(\theta) = u(\theta) \mathrm{tg}^\beta(\theta/2) \qquad (0 < \theta < \pi)$$

where $\beta = -(q/p)$; then, by Riesz's theorem,

$$(2.4) \qquad \int_0^\pi |V(\theta)|^p d\theta \leqq K(p) \int_0^\pi |U(\theta)|^p d\theta = K(p) \int_0^{\pi-\delta} |U(\theta)|^p d\theta.$$

Writing $w(\theta) = v(\theta) \mathrm{tg}^\beta(\theta/2) - V(\theta)$, we have, by Riesz's theorem,

$$(2.5) \qquad \int_0^\pi |v(\theta)|^p \theta^{-q} d\theta \leqq \int_0^{\pi/2} |v(\theta)|^p \theta^{-q} \sec^2(\theta/2) d\theta + \pi^p \int_0^\pi |u(\theta)|^p d\theta.$$

The integral

$$\int_0^{\pi/2} |v(\theta)|^p \theta^{-q} \sec^2(\theta/2) d\theta$$

is not greater than the sum

$$K(p,q) \left(\int_0^{\pi/2} |w(\theta)|^p \sec^2(\theta/2) d\theta + \int_0^{\pi/2} |V(\theta)|^p \sec^2(\theta/2) d\theta \right).$$

It is therefore, by (2.4) and (2.5), sufficient to prove that

$$(2.6) \qquad \int_0^\pi |w(\theta)|^p \sec^2(\theta/2) d\theta \leqq K \int_0^\pi |U(\theta)|^p \sec^2(\theta/2) d\theta.$$

Writing

$$\xi = \operatorname{tg}(\theta/2), \quad \eta = \phi/2, \quad H = H(\xi, \eta) = \frac{\xi(\eta^\beta - \xi^\beta)}{\eta^\beta(\eta^2 - \xi^2)},$$

then

$$\frac{\sin\theta}{\cos\phi - \cos\theta} = \frac{\xi}{\xi^2 - \eta^2} \sec^2(\phi/2)$$

and

$$w(\theta) = (1/\pi) \int_0^\pi \frac{\xi}{\xi^2 - \eta^2} \; (u(\phi) \operatorname{tg}^\beta(\theta/2) - U(\phi)) \sec^2(\phi/2) d\phi$$

$$= (1/\pi) \int_0^\pi - HU(\phi) \sec^2(\phi/2) d\phi.$$

Hence we have

$$\int_0^\pi | w(\theta) |^p \sec^2(\theta/2) d\theta$$

$$= \int_0^\pi | w(\theta) |^{p-1} \sec^2(\theta/2) \left| (1/\pi) \int_0^\pi HU(\phi) \sec^2(\phi/2) d\phi \right|,$$

$$\leq (1/\pi) \int_0^\pi \int_0^\pi | w(\theta) |^{p-1} [(1 + \xi^2) \; (1 + \eta^2) \; | H | \; (\xi/\eta)^{1/p}]^{1/p'}$$

$$\cdot | U(\phi) | \; [(1 + \xi^2)(1 + \eta^2) \; | H | \; (\eta/\xi)^{1/p'}]^{1/p} d\theta d\phi,$$

where $1/p + 1/p' = 1$. Hölder's inequality gives

$$(2.7) \qquad \int_0^\pi | w(\theta) |^p \sec^2(\theta/2) d\theta \leq (1/\pi) J_1^{1/p'} J_2^{1/p}.$$

Putting $\eta/\xi = t$, we find that

$$\int_0^\pi (\xi/\eta)^{1/p} | H | \sec^2(\phi/2) d\phi = 2 \int_0^\infty t^{-1/p + q/p} \left| \frac{t^\beta - 1}{t^2 - 1} \right| \; dt = C,$$

is convergent, since $-p < q - 1 < p$; and that

$$\int_0^\pi (\eta/\xi)^{1/p'} | H | \sec^2(\theta/2) d\theta = C.$$

Hence

$$J_1 \leq C \int_0^\pi | w(\theta) |^p \sec^2(\theta/2) d\theta \quad \text{and} \quad J_2 \leq C \int_0^\pi | U(\phi) |^p \sec^2(\phi/2) d\phi.$$

This, combined with (2.7), establishes (2.6):

$$\int_0^\pi | w(\theta) |^p \sec^2(\theta/2) d\theta \leq C^p \int_0^\pi | U(\phi) |^p \sec^2(\phi/2) d\phi.$$

Lemma 1 is thus proved.

3. Lemmas concerning power series.

LEMMA 2. *Suppose that* $q \geqq 0$, *that the odd function* $\psi(t)$ *is the imaginary part of the boundary function*

$$g(\theta) = F(e^{i\theta}) = \phi(t) + i\psi(t),$$

where $F(z)$ *is regular in the unit circle. Then* (1.6) *with* (1.1) *implies*

$$(3.1) \qquad \int_{-\pi}^{\pi} |F^{(j)}(re^{i\theta})|^p |\theta|^{-q} d\theta = O((1-r)^{kp-jp}),$$

as $r \to 1-0$, *where* $F^{(j)}(z)$ *denotes the j-th derivative of* $F(z)$.

The condition (1.6), with $q \geqq 0$, implies $\psi(t) \in L^p(0,\pi)$.[9] It follows from Riesz's inequality that $\phi(t) \in L^p(0,\pi)$. Hence $g(\theta)$ belongs to $L^p(-\pi,\pi)$. Then,[10] if $0 < r < 1$,

$$(2\pi/j!)F^{(j)}(re^{i\theta}) = \int_{-\pi}^{\pi} \frac{g(t)e^{it}dt}{(e^{it}-re^{i\theta})^{1+j}} = e^{-ji\theta} \int_{-\pi}^{\pi} \frac{g(\theta+t)e^{it}dt}{(e^{it}-r)^{1+j}},$$

$$0 = e^{-ji\theta} \int_{-\pi}^{\pi} \frac{g(\theta+t)e^{jit}dt}{(1-re^{it})^{1+j}} = e^{-ji\theta} \int_{-\pi}^{\pi} \frac{g(\theta-t)e^{it}dt}{(e^{it}-r)^{1+j}}.$$

Hence

$$F^{(j)}(re^{i\theta}) = \frac{j! \, e^{-ji\theta}}{2\pi} \int_{-\pi}^{\pi} \frac{[g(\theta+t)-g(\theta-t)]e^{it}dt}{(e^{it}-r)^{1+j}},$$

and

$$\left(\int_{-\pi}^{\pi} |F^{(j)}(re^{i\theta})|^p |\theta|^{-q} d\theta \right)^{1/p}$$

$$\leqq (j!/2\pi) \int_{-\pi}^{\pi} \frac{dt}{|e^{it}-r|^{1+j}} \left(\int_{-\pi}^{\pi} \frac{|g(\theta+t)-g(\theta-t)|^p d\theta}{|\theta|^q} \right).$$

Evidently, we may assume that $F(0) = 0$. The even function $\psi(\theta+t) - \psi(\theta-t)$ of θ is the real part of

$$-i(g(\theta+t)-g(\theta-t)).$$

The conjugate $\phi(\theta-t) - \phi(\theta+t)$ satisfies, by Lemma 1, the relation

$$\int_0^{\pi} |\phi(\theta+t) - \phi(\theta-t)|^p \theta^{-q} d\theta = O(|t|^{pk}).$$

Accordingly we have

$$(3.2) \qquad \int_{-\pi}^{\pi} |g(\theta+t) - g(\theta-t)|^p |\theta|^{-q} d\theta = O(|t|^{pk}).$$

It follows that

[9] Hardy and Littlewood [4].

[10] F. and M. Riesz [9]

$$\left(\int_{-\pi}^{\pi}|F^{(j)}(re^{i\theta})|^{p}\cdot|\theta|^{-q}\,d\theta\right)^{1/p}=O\left(\int_{-\pi}^{\pi}\frac{|t|^{k}\,dt}{|e^{it}-r|^{1+j}}\right)=O((1-r)^{k-j}).$$

This completes the proof.

From the above argument, we can state the following proposition:

LEMMA 3. *If $q \geqq 0$, $F(z) = \Sigma c_n z^n$ is regular in $|z| < 1$, and the boundary function*

$$(3.3) \qquad\qquad g(\theta) = F(e^{i\theta})$$

satisfies the relation (3.2) *with* (1.1), *then* (3.1) *is true.*

If $q = 0$, the converse of Lemma 3 is valid:

LEMMA 4. *If $p > 1$, $0 < k < 1$, $z = e^{i\theta}$, $F(z)$ is regular in the unit circle and, for some j,*

$$(3.4) \qquad\qquad \int_{-\pi}^{\pi}|F^{(j)}(z)|^{p}\,d\theta = O((1-r)^{kp-jp}),$$

as $r \to 1 - 0$, then the boundary function (3.3) *satisfies the relation*

$$(3.5) \qquad\qquad \int_{-\pi}^{\pi}|g(\theta+t)-g(\theta-t)|^{p}\,d\theta = O(|t|^{pk}).$$

If $j > 1$, then, writing $w = \rho e^{i\theta}$, we have

$$\left(\int_{-\pi}^{\pi}|F^{(j-1)}(z)|^{p}\,d\theta\right)^{1/p} = \left(\int_{-\pi}^{\pi}d\theta\left|\int_{0}^{z}(F^{(j)}(w)+z^{-1}F^{(j-1)}(0))\,dw\right|\right)^{1/p}$$

$$\leqq \int_{0}^{r}\left(\left(\int_{-\pi}^{\pi}|F^{(j)}(w)+z^{-1}F^{(j-1)}(0)|^{p}\,d\theta\right)^{1/p}\,d\rho\right.$$

$$= O\left(\int_{0}^{r}(1-\rho)^{k-j}d\rho\right) = O((1-r)^{k-j+1}).$$

The proposition is thus reduced to the case $j = 1$, which is a known theorem.[11]

LEMMA 5. *If the function*

$$F(z) = \sum_{n=0}^{\infty}c_n z^n \qquad (z = re^{i\theta})$$

is regular in the unit circle, and the relation

$$\int_{-\pi}^{\pi}\frac{|F^{(j)}(z)|^{p}\,d\theta}{|1-z|^{q}} = O((1-r)^{kp-jp}), \qquad (r \to 1-0)$$

holds for some j, then it is true for every j, where $p > 1$, $0 < k < 1$, $0 \leqq q \leqq 1 - k$, and $j = 1, 2, \cdots$.

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[11] Hardy and Littlewood [2], Theorem 3.

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In fact, letting $j > 1$, $w = \rho e^{i\vartheta}$, we have

$$\left(\int_{-\pi}^{\pi} \frac{|F^{(j-1)}(z)|^p \, d\theta}{|1-z|^q}\right)^{1-r} = \left(\int_{-\pi}^{\pi} \frac{d\theta}{|1-z|^q} \left| \int_0^z (F^{(j)}(w) + z^{-1}F^{(j-1)}(0)) \, dw \right.\right.$$

$$\leq \int_0^r \left(\frac{1-\rho}{1-r}\right)^q \left(\int_{-\pi}^{\pi} \frac{|F^{(j)}(w)|^p \, d\theta}{|1-w|^q}\right)^{1/p} d\rho + O(1-r)^{-q}$$

$$= O((1-r)^{k-j+1}).$$

On the other hand, writing $z = re^{i\theta}$, $w = \sqrt{r}e^{i\phi}$, we have

$$F^{(j+1)}(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{F^{(j)}(w) w \, d\phi}{(w-z)^2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{F^{(j)}(we^{i\theta}) we^{-i\theta} d\phi}{(w-r)^2} .$$

Observing that

$$\left| \frac{1-we^{i\theta}}{1-z} \right|^q = O(1) + O\left(\frac{|\phi|^q}{(1-r)^q}\right),$$

we obtain

$$\left(\int_{-\pi}^{\pi} \frac{|F^{(j+1)}(z)|^p \, d\theta}{|1-z|^q}\right)^{1/p} \leq \int_{-\pi}^{\pi} \frac{d\phi}{|w-r|^2} \left(\int_{-\pi}^{\pi} \left| \frac{1-we^{i\theta}}{1-z} \right|^q \frac{|F^{(j)}(we^{i\theta})|^p \, d}{|1-we^{i\theta}|^q}\right.$$

$$= \int_{-\pi}^{\pi} \frac{O((1-r)^{k-j}) \, d\phi}{|w-r|^2} + \int_{-\pi}^{\pi} O((1-r)^{k-j-q/p}) \frac{|\phi|^{q/p} \, d\phi}{|w-r|^2}$$

$$= O((1-r)^{k-j-1}).$$

This establishes the lemma.

4. Summability of power series. We write $(\alpha)_0 = 1$, $(-1)_n = 0$ for $n > 0$,

$$(\alpha)_n = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)},$$

and

$$(\alpha)_n \sigma_n^a = \sum_{\nu=0}^n (\alpha)_{n-\nu} c_\nu. \quad \tau_n^a = n(\sigma_n^a - \sigma_{n-1}^a) = \frac{1}{(\alpha)_n} \sum_{\nu=0}^n (\alpha-1)_{n-\nu} \nu c_\nu,$$

where $\alpha > -1$. The series Σc_n is summable $|C, \alpha|$ if $\Sigma n^{-1}\tau_n^a$ converges absolutely.

Proof of Theorem 4. We have

$$\Sigma (\alpha)_n \tau_n^a z^n = zF'(z)(1-z)^{-a}.$$

Hence

$$(4.1) \quad \sum_{n=0}^{\infty} \tau_n^a z^{n+a} = \int_0^z (z-w)^a (d/dw)(wF'(w)(1-w)^{-a}) \, dw$$

$$= I_1 + I_2 + I_3,$$

where $z = re^{i\theta}$, $w = \rho e^{i\theta}$, $0 \leqq \rho \leqq r < 1$; and

$$I_1 = \int_0^z (z-w)^a (1-w)^{-a} F'(w)\, dw,$$

$$I_2 = \int_0^z (z-w)^a w (1-w)^{-a} F''(w)\, dw,$$

$$I_3 = \int_0^z (z-w)^a \alpha w (1-w)^{-a-1} F'(w)\, dw.$$

For the proof of the theorem we can assume that $p \leqq 2$. In fact, if $p > 2$, taking η greater than $1 - 2k$ and less than $(p + 2q - 2)/p$, then Hölder's inequality gives

$$\left(\int_{-\pi}^{\pi} \frac{|F^{(j)}(z)|^2\, d\theta}{|1-z|^{\eta}} \right)^{\frac{1}{2}} \leqq \left(\int_{-\pi}^{\pi} |1-z|^Q\, d\theta \right)^{(p-2)/2p} \left(\int_{-\pi}^{\pi} \frac{|F^{(j)}(z)|^p\, d\theta}{|1-z|^q} \right)^{1/p}.$$

This is equal to $O((1-r)^{k-j})$, since $Q = \dfrac{2q - \eta p}{p-2} > -1$. Accordingly, it is enough to prove the theorem for

$$1/p - k < \alpha < q/p,$$

by a theorem of Kogbetliantz.[12] Hence

$$\left(\int_{-\pi}^{\pi} |I_2|^p\, d\theta \right)^{1/p} = \left(\int_{-\pi}^{\pi} d\theta \left| \int_0^z (z-w)^a w (1-w)^{-a} F''(w)\, dw \right|^p \right)^{1/p}$$

$$\leqq 2 \int_0^r (r-\rho)^a \left(\int_{-\pi}^{\pi} \frac{|F''(w)|^p\, d\theta}{|1-w|^q} \right)^{1/p} d\rho$$

$$= O\left(\int_0^r (r-\rho)^a (1-\rho)^{k-2} d\rho \right),$$

by Lemma 5. Observing that $\alpha + 1 > 0$, integration by parts gives

$$\int_0^r (r-\rho)^a (1-\rho)^{k-2} d\rho = \frac{r^{a+1}}{1+\alpha} + \frac{2-k}{1+\alpha} \int_0^r (r-\rho)^{a+1} (1-\rho)^{k-3} d\rho$$

$$\leqq \frac{1}{1+\alpha} + \frac{2-k}{1+\alpha} \int_0^r (1-\rho)^{a+k-2} d\rho.$$

It follows that

(4.2) $$\int_{-\pi}^{\pi} |I_j|^p\, d\theta = O((1-r)^{ap+kp-p})$$

for $j = 2$. (4.2) is also true for $j = 1$, since the above argument is applicable to I_1. Further, observing that

$$(1-w)^{-a-1} = O((1-\rho)^{-1} |1-w|^{-q/p}),$$

[12] Kogbetliantz [8].

we have

$$(\int_{-\pi}^{\pi} | I_3 |^p d\theta)^{1/p} = O(\int_0^r (r-\rho)^a (1-\rho)^{-1} \cdot (1-\rho)^{k-1} d\rho).$$

Hence (4.2) is true for $j = 3$. From (4.1) and (4.2), we obtain

$$(4.3) \qquad \mu(\rho) \equiv (\int_{-\pi}^{\pi} | \sum_1^\infty \tau_n^a z^n |^p d\theta)^{1/p} = O((1-r)^{a+k-1}).$$

The expression $\mu(p)$ is a non-decreasing function of p. Hence

$$\mu(\min(p, 2)) = O((1-r)^{a+k-1}).$$

Write

$$P = \frac{\min(p, 2)}{\min(p, 2) - 1};$$

then Hausdorff's inequality gives

$$\sum | \tau_n^a r^n |^P = O((1-r)^{aP+kP-P}).$$

Hence

$$\sum_1^n | \tau_\nu^a |^P = O(n^{P-aP-kP}),$$

on taking $r = 1 - 1/n$. This implies the absolute convergence of the series $\sum n^{-1} \tau_n^a$. Theorem 4 is thus proved.

COROLLARY. If $p > 1$, $0 < kp \leq 1$, $\alpha > \alpha_0$, $z = re^{i\theta}$ and $F(z) = \sum c_n z^n$, then (1.11) implies the summability $| C, \alpha |$ of $\sum c_n e^{ni\theta}$.

For the proof, we may suppose that $\theta = 0$. Let $1 - pk < q < 1$, and write $H = \sqrt{(1-r)^2 + \phi^2}$, then

$$\int_0^{1-r} H^{-q-2} \phi d\phi = O((1-r)^{-q}), \qquad \int_{1-r}^{\pi} H^{-q-2} \phi^2 d\phi = O(1),$$

as $r \to 1 - 0$. It follows that

$$\int_0^{\pi} | F'(z) |^p H^{-q} d\phi = G(r, \pi)(H(\pi))^{-q} + q(\int_0^{1-r} + \int_{1-r}^{\pi}) G(r, \phi) H^{-q-2} \phi d\phi$$

$$= O((1-r)^{pk-p}) + O(G(r, 1-r)(1-r)^{-q}) = O((1-r)^{pk-p}).$$

The same argument applies to $\int_0^{-\pi} | F' |^p H^{-q} d\phi$. Hence

$$\int_{-\pi}^{\pi} | F'(z) |^p H^{-q} d\phi = O((1-r)^{pk-p}).$$

This implies (1.9), and the conclusion follows from Theorem 4.

THEOREM 5.   *If* $p > 1$, $0 < k < 1$, $\alpha > \alpha_0$, *and the function* $F(z)$ $= \Sigma c_n z^n$ $(z = re^{i\theta})$ *satisfies the relation*

$$(4.4) \qquad \int_{-\pi}^{\pi} |F^{(j)}(z)|^p\, d\theta = O((1-r)^{kp-jp})$$

for some $j > 0$, *as* $r \to 1 - 0$, *then the series* $\Sigma c_n e^{ni\theta}$ *is summable* $|\, C, \alpha\,|$ *at every point* $e^{i\theta}$ *of the unit circle where the function* $F(z)$ *is regular.*

In view of Theorem 4, we have only to prove the theorem for the case $pk \leqq 1$.

We may without loss of generality take the point in question to be $z = 1$. Then 1 is a zero of the derivative of the function

$$G(z) = c_0 + (c_1 - F'(1))z + \cdots = F(z) - F'(1)z.$$

It follows that $G'(z) = O(|\,1-z\,|)$, as $z \to 1$, and

$$\int_{-\pi}^{\pi} \frac{|\,G'(z)\,|^p\, d\theta}{|\,1-z\,|} = O((1-r)^{pk-p}),$$

by (4.4) and Lemma 5. Thus, by Theorem 4, the series $c_0 + (c_1 - F'(1))$ $+ c_2 + \cdots$ is summable $|\, C, \alpha\,|$, with $\alpha > \alpha_0$. This completes the proof of Theorem 5.

5.   Proof of Theorem 3.   Let $\Sigma A_n(t)$ be the Fourier series of $f(t)$, and write $z = re^{i\theta}$,

$$\Sigma A_n(x) z^n = F(z),$$

then the odd function $\psi(t)$ is the imaginary part of the boundary function of $F(z)$. In virtue of Lemma 2, (3.1) holds true. *A fortiori*, (1.9) is true. It follows from Theorem 4 that $\Sigma A_n(x)$ is summable $|\, C, \alpha\,|$, whenever $\alpha > \alpha_0$.

To complete the proof of the theorem, we require the following lemmas.

LEMMA 6.   *If the series* ΣA_n *is summable* (C) *and*

$$\sum_{\nu=1}^{n} |\,\nu A_\nu\,|^p = O(n)$$

where $p > 1$, *then* ΣA_n *is summable* $(C, 1/p - 1 + \delta)$ *for every positive* δ. This is a known theorem.[13]

LEMMA 7.   *If* $\alpha > -1$, $\beta > -1$, $\alpha + \beta > -1$, *and the series* $\Sigma n^{-1}\tau_n^\alpha$ *with*

_____

[13] Hardy and Littlewood [2], Lemma 4.

$$\tau_n{}^a = (1/(\alpha)_n) \sum_{1}^{n} (\alpha - 1))_{n-\nu}\nu A_\nu$$

is summable (C, β), *then the series* ΣA_n *is summable* $(C, \alpha + \beta)$.

This is known as Hausdorff's theorem.[14]

In the equation (4.1), let us set $\alpha = 1/p - k$ and

$$\tau_n{}^a = (1/(\alpha)_n) \sum_{\nu=1}^{n} (\alpha - 1)_{n-\nu}\nu A_\nu(x) ;$$

then from the proof of Theorem 4, we have (4.3), i. e.

$$\int_{-\pi}^{\pi} | \sum \tau_n{}^a z^n |^p d\theta = O((1 - r)^{1-p}).$$

If $p \leq 2$, then by Hausdorff's inequality,

$$\sum_{1}^{\infty} | \tau_n{}^a r^n |^{p/(p-1)} = O(1/(1 - r)).$$

Taking $r = 1 - 1/n$, we obtain

$$\sum_{\nu=1}^{n} | \tau_\nu{}^a |^{p/(p-1)} = O(n).$$

The series $\Sigma n^{-1} \tau_n{}^a$ then satisfies the conditions of Lemma 6, and so it is summable $(C, (p-1)/p - 1 + \delta)$ for $\delta > 0$. Therefore the series $\Sigma A_n(x)$, by Lemma 7, is summable (C, β), where

$$\beta = ((\beta - 1)/p - 1 + \delta) + (1/p - k) = \delta - k.$$

If $p > 2$, take a number η such that

$$1 - 2k < \eta < (p + 2q - 2)/p ;$$

then by Hölder's inequality,

$$\int_{0}^{\pi} | \psi(t + h) - \psi(t - h) |^2 t^{-\eta} dt$$

$$\leq (\int_{0}^{\pi} t^Q dt)^{(p-2)/p} (\int_{-\pi}^{\pi} | \psi(t + h) - \psi(t - h) |^p t^{-q} dt)^{2/p},$$

where

$$Q = \frac{2q - \eta p}{p - 2} > \frac{2 - p}{p - 2} = -1.$$

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14 Hausdorff [6].

It follows from (1.6) that

$$\int_0^\pi |\psi(t+h) - \psi(i-h)|^2 t^{-\eta} dt = O(h^{2k}),$$

with $\eta + 2k > 1$. This completes the proof.

6.   An extension of a theorem of Zygmund. We are now in a position to prove Theorem 2, but our method of proof leads us to establish the more general result which follows.

THEOREM 6.   *Let* $f(\theta) \sim \Sigma A_n(\theta)$, $1 \leqq p_1 \leqq 2 \leqq p_2$, $0 < k_j \leqq 1$, *and*

(6.1) $1 \leqq \min(k_1 p_1, k_2 p_2) < \max(k_1 p_1, k_2 p_2).$

If (1.4) *holds for* $p = p_j$, $k = k_j$ $(j = 1, 2)$, *then* $\Sigma A_n(\theta)$ *is summable* $|C, \alpha|$, *when*

(6.2) $\alpha > \tfrac{1}{2} - \kappa,$

and is summable (C, β), *when*

(6.3) $\beta > -\kappa,$

where

$$\kappa = \frac{k_1 p_1(p_2 - 2) + k_2 p_2(2 - p_1)}{2(p_2 - p_1)}.$$

We have $\kappa > \tfrac{1}{2}$, by (6.1). If $p_1 = k_1 = 1$, or $p_2 \to \infty$ and $p_1 k_1 = 1$, then we obtain theorems including Hardy-Littlewood's extensions [15] of Zygmund's theorem of absolute convergence of Fourier series. Let $p_1 = k_1 = 1$ and $p_2 \to \infty$; then Theorem 6 is reduced to Theorem 2, since any function $f(\theta)$ of bounded variation is characterized by (1.4) with $p = k = 1$.

Writing $\Delta = |f(\theta + h) - f(\theta - h)|$, we have

(6.4) $\int \Delta^2 d\theta \leqq (\int \Delta^{p_1} d\theta)^{(p_2-2)/(p_2-p_1)} (\int \Delta^{p_2} d\theta)^{(2-p_1)/(p_2-p_1)}.$

From the condition (1.4) with $p = p_j$, $k = k_j$, and (6.4), we obtain

(6.5) $\int_{-\pi}^\pi |f(\theta + h) - f(\theta - h)|^2 d\theta = O(h^{2\kappa}).$

This is, after Hardy and Littlewood, the convexity property of the relation (1.4). The required results are immediate consequences of Theorem 3, since $2\kappa > 1$.

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EULER TRANSFORMATIONS.*

By Ralph Palmer Agnew.

1. Introduction. A series $u_0 + u_1 + \cdots$, and its sequence $s_0, s_1, \cdots$ of partial sums, are said to be summable to σ by the Euler transformation (or method of summability) $E(r)$ of order r, r being a complex constant, if $\sigma_n \to \sigma$ as $n \to \infty$ where

$$E(r) \qquad \sigma_n \equiv \sigma_n(r) = \sum_{k=0}^{n} \binom{n}{k} r^k (1-r)^{n-k} s_k.$$

The statement that the sequence σ_n is the $E(r)$ transform of the sequence s_n will be abbreviated in the form $\sigma_n = E(r)s_n$. It is well known that the family of Euler methods $E(r)$ for which r is real and $0 < r < 1$ is a consistent family of regular methods of summability; the theory of this family has been well developed. It is the object of this paper to establish fundamental properties of methods $E(r)$ for the general case in which r is complex.

The transformation $E(r)$ has, for each fixed r, the form

$$(1.1) \qquad \sigma_n = \sum_{k=0}^{n} a_{nk} s_k$$

where

$$(1.2) \qquad a_{nk} = \binom{n}{k} r^k (1-r)^{n-k}.$$

We observe that

$$(1.3) \qquad \sum_{k=0}^{n} |a_{nk}| = (|r| + |1-r|)^n \qquad (n = 0, 1, 2, \cdots),$$

and that this sequence is bounded if and only if $0 \leqq r \leqq 1$. By the well known Silverman-Toeplitz Theorem, (1.1) is regular (such that the existence of $\lim s_n$ implies $\lim \sigma_n = \lim s_n$) if and only if

$$(1.41) \qquad \sum_{k=0}^{n} |a_{nk}| < M \qquad (n = 0, 1, 2, \cdots),$$

$$(1.42) \qquad \lim_{n \to \infty} a_{nk} = 0 \qquad (k = 0, 1, 2, \cdots),$$

$$(1.43) \qquad \lim_{n \to \infty} \sum_{k=0}^{n} a_{nk} = 1,$$

M being a constant independent of n. Using these conditions, it is easy to establish the known result [1] that $E(r)$ is regular if and only if r is real and $0 < r \leq 1$. The transformation $E(1)$ is the identity. The transformation $E(0)$ is a trivial non-regular transformation which transforms the sequence $s_0, s_1, s_2, \cdots$ into the sequence $s_0, s_0, s_0, \cdots$; this transformation plays no interesting role in our work and henceforth we assume that all numbers p, q, r which represent orders of Euler transformations are different from 0.

Corresponding to each pair p and q of complex constants, the $E(p)$ transform of the $E(q)$ transform of a sequence s_n is σ_n where

$$(1.5) \qquad \sigma_n = \sum_{m=0}^{n} \binom{n}{m} p^m (1-p)^{n-m} \sum_{k=0}^{m} \binom{m}{k} q^k (1-q)^{m-k} s_k$$

$$= \sum_{k=0}^{n} \binom{n}{k} (pq)^k \sum_{m=k}^{n} \binom{n-k}{m-k} (p-pq)^{m-k} (1-p)^{n-m} s_k$$

$$= \sum_{k=0}^{n} \binom{n}{k} (pq)^k \sum_{m=0}^{n-k} \binom{n-k}{m} (p-pq)^m (1-p)^{n-k-m} s_k$$

$$= \sum_{k=0}^{n} \binom{n}{k} (pq)^k (1-pq)^{n-k} s_k.$$

Thus the product transformation $E(p)E(q)$ is identical with the transformation $E(pq)$; that is,

$$(1.6) \qquad\qquad E(p)E(q) = E(pq).$$

It follows that $E(p)$ and $E(q)$ commute.[2] Setting $q = p^{-1}$ gives $E(p)E(p^{-1}) = E(1)$. Thus the inverse $E^{-1}(p)$ of $E(p)$ is $E(p^{-1})$, that is

$$(1.9) \qquad\qquad E^{-1}(p) = E(p^{-1}).$$

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[1] See Knopp [8], p. 246 and Hurwitz [6], p. 22. The parameter r of the present paper is the reciprocal of that of Hurwitz.

[2] The whole family of transformations $E(r)$ belongs to the class of transformations, studied by Hurwitz and Silverman [7] and by Hausdorff [5], which commute with the arithmetic mean transformation and with each other. For each complex r, $E(r)$ has the Hurwitz-Silverman form

$$(1.7) \qquad \sigma_n = \sum_{k=0}^{n} \binom{n}{k} \{ \sum_{j=k}^{n} (-1)^{j-k} \binom{n-k}{j-k} \lambda(j) \} s_k$$

where $\lambda(j) = r^j$. It is only when r is real and $0 < r \leq 1$ that $E(r)$ can be written in the regular Hausdorff form

$$(1.8) \qquad \sigma_n = \int_0^1 \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} s_k d\chi(t),$$

and in this case $\chi(t) = 0$ or 1 according as $t < r$ or $t > r$. When $0 < r \leq 1$, the function λ of the Hurwitz-Silverman formula is the moment function $\lambda(z) = \int_0^1 t^z d\chi(t)$ of the mass function χ of the Hausdorff formula. For relations among regular Hurwitz-Silverman-Hausdorff methods, see Garabedian, Hille, and Wall [4].

A transformation A_2 is said to include a transformation A_1, and one writes $A_2 \supset A_1$, if each sequence summable A_1 is also summable A_2, the two values being equal. By a well known criterion for inclusion, $E(p) \supset E(q)$ if, and only if, $E(p) E^{-1}(q)$ is regular. Since $E(p) E^{-1}(q) = E(p/q)$, it follows that $E(p) \supset E(q)$ *if and only if a real number* θ *exists such that* $0 < \theta \leqq 1$ *and* $p = \theta q$. Thus an inclusion relation subsists between two transformations $E(p)$ and $E(q)$ if and only if the complex numbers p and q representing the orders lie on the same half-line radiating from the origin in the complex plane. The transformation represented by the point nearer the origin is the stronger of the two.

We note, in particular, that if $r > 1$, then $E(r) \subset E(1)$; this means that if $r > 1$, then each series or sequence summable $E(r)$ must be convergent. Some such methods of summability have been studied extensively, notably the Cesàro methods C_r of orders r for which $-1 < \mathcal{R} r < 0$.

2. A necessary condition for summability $E(r)$. If $r \neq 0$ and a sequence s_k is summable $E(r)$, then the transform σ_n must be bounded, say $|\sigma_n| \leqq M$, $(n = 0, 1, 2, \cdots)$, and therefore

$$(2.1) \quad |s_n| = \Big| \sum_{k=0}^{n} \binom{n}{k} (1/r)^k (1 - 1/r)^{n-k} \sigma_k \Big|$$

$$\leqq M \sum_{k=0}^{n} \binom{n}{k} |1/r|^k |1 - 1/r|^{n-k} = M(|1/r| + |1 - 1/r|)^n.$$

It follows from this necessary condition for $E(r)$ summability that if the sequence s_k is summable $E(r)$, then the power series $\Sigma s_k z^k$ must have radius of convergence at least $(|r^{-1}| + |1 - r^{-1}|)^{-1}$.

If a series $u_0 + u_1 + \cdots$ has partial sums $s_0, s_1, \cdots$ so that, when s_{-1} is defined to be 0,

$$u_n = s_n - s_{n-1} \qquad\qquad (n = 0, 1, 2, \cdots),$$

and if $|s_n| \leqq M_1 R^n$ where M_1 and R are positive constants, then the crude estimate

$$|u_n| \leqq |s_n| + |s_{n-1}| \leqq M_1 R^n + M_1 R^{n-1}$$

shows that $|u_n| < M_2 R^n$ where $M_2 = M_1(1 + R^{-1})$. This fact and the inequality (2.1) show that if Σu_n is summable $E(r)$, then

$$(2.2) \qquad\qquad |u_n| \leqq M_2(|r|^{-1} + |1 - r^{-1}|)^n$$

and the radius of convergence of $\Sigma u_n z^n$ is at least $(|r^{-1}| + |1 - r^{-1}|)^{-1}$.

3. Summability of the sequence z^k. It is possible to draw conclusions concerning $E(r)$, for complex as well as real values of r, by considering summability of the sequence z^k. For each r, the $E(r)$ transform $\sigma_n(r, z)$ of the sequence z^k is

$$(3.1) \qquad \sigma_n(r, z) = (1 - r + rz)^n.$$

If $z = 1$, the sequence z^k is summable $E(r)$ to 1 for each r. If $z \neq 1$, then the sequence is summable $E(r)$ if and only if $|1 - r + rz| < 1$, that is

$$(3.2) \qquad |z - (1 - 1/r)| < 1/|r|.$$

It follows that, when $r \neq 0$ is fixed, the set of values of z for which the sequence z^k is summable E_r consists of the point $z = 1$ and the interior of the circle $C(r)$ with center at the point $(1 - r^{-1})$ and radius $|r|^{-1}$. This circle $C(r)$ passes through the point $z = 1$. In particular, the interior of the circle $|z - 2| < 1$, where the sequence z^k is summable $E(-1)$ to 0, contains no points in common with the interior of the unit circle $|z| < 1$ where the sequence converges to 0. The circle $|z - 14/15| < 1/15$ in which z^k is summable $E(15)$ to 0 is a small subset of the circle of convergence; and the circle $|z + 14| < 15$ in which the sequence is summable $E(1/15)$ to 0 includes and is larger than the circle of convergence.

It is well known (Hurwitz [6]) that the Borel exponential method B includes $E(r)$ when $0 < r \leq 1$. Hence also $B \supset E(r)$ when $r > 0$. If r is a complex number not both real and ≥ 0, then the center of the circle $C(r)$ in which z^k is summable $E(r)$ to 0 does not lie on the segment $x < 1$ of the real axis and accordingly z^k must be summable $E(r)$ to 0 for some z for which $\mathcal{R} z > 1$. Since z^k is not summable B when $\mathcal{R} z > 1$, this implies that B does not include $E(r)$. Thus we obtain the following theorem:

THEOREM 3.3. *The Borel exponential method B includes $E(r)$ if and only if r is real and positive.*

It is known (Morse [10], p. 281) that the LeRoy method LR includes $E(r)$ when $0 < r \leq 1$ and hence that $LR \supset E(r)$ when $r > 0$. Let $\mathcal{R} r < 0$. Then the center of the circle $C(r)$ has real part greater than 1. This implies the existence of a real number $z_0 > 1$ such that z_0^n is summable $E(r)$. It follows that if $\mathcal{R} r < 0$, then neither the LeRoy method nor any other totally regular method of summability can include $E(r)$. The question whether $LR \supset E(r)$ when $\mathcal{R} r \geq 0$ and r is not real is left open; the method of Morse [10] does not apply to this case.

The following theorem shows that, except for those values of r for which

each sequence summable $E(r)$ must be convergent, there is no inclusion relation between $E(r)$ and a regular Cesàro method $C(q)$.

THEOREM 3.4. *If r and q are complex numbers for which r is not both real and $\geqq 1$ while $\Re q > 0$, then neither of the methods of summability $E(r)$ and $C(q)$ includes the other.*

Under the hypotheses on r, the series Σz^n and the sequence $(1-z^{n+1})/(1-z)$ of partial sums are summable $E(r)$ for some values of z outside the circle of convergence. Using the well known fact that Cesàro methods are ineffective outside circles of convergence, we see that $C(q)$ does not include $E(r)$. If $\Re q > 0$ and r is not in the real interval $0 \leqq r \leqq 1$, then $C(q)$ evaluates all convergent sequences whereas $E(r)$ does not; hence in this case $E(r)$ does not include $C(q)$. Obviously $E(0)$ does not include $C(q)$. It was shown by Knopp [8], pp. 251-253, that the sequence

$$(3.41) \qquad\qquad 0, 1, 1, 1, 0, 0, 0, 0, 0, 1, \cdots$$

in which the successive groups of 0's and 1's contain respectively $1, 3, 5, 7, \cdots$ elements, is summable $C(1)$ to $\frac{1}{2}$ and is nonsummable $E(p)$ when $p = 2^{-1}$, $2^{-2}, 2^{-3}, \cdots$. Since (see, for example, Kogbetliantz [9], p. 24) a bounded sequence summable $C(1)$ is summable $C(q)$ when $\Re q > 0$, it follows that (3.41) is summable $C(q)$. If $0 < r < 1$, then a positive integral exponent j can be chosen such that $2^{-j} < r$ and $E(2^{-j}) \supset E(r)$, and it follows that (3.41) cannot be summable $E(r)$. Therefore, if $\Re q > 0$ and $0 < r < 1$, $E(r)$ cannot include $C(q)$. This proves Theorem 3.4.

4. Omission and adjunction of elements. A transformation A is said to *permit omission of elements* if summability of $s_0, s_1, s_2, \cdots$ implies summability of the sequence $s_1, s_2, \cdots$ to the same value. Let $\sigma_n(r)$ denote, as above, the $E(r)$ transform of $s_0, s_1, s_2, \cdots$; and let $\tau_n(r)$ denote the $E(r)$ transform of $s_1, s_2, s_3, \cdots$. It is easy to show, by simplifying the right member, that

$$(4.1) \qquad \tau_n(r) = (1 - r^{-1})\sigma_n(r) + r^{-1}\sigma_{n+1}(r).$$

Hence obviously $\sigma_n(r) \to \sigma$ implies $\tau_n(r) \to \sigma$. This gives

THEOREM 4.2. *If r is a complex number not 0, then $E(r)$ permits omission of elements.*

This result was obtained by Knopp [8], p. 233, for the case $r = 1/2$ and by Silverman [13], p. 382, for the case $0 < r \leqq 1$.

A transformation A is said to *permit adjunction of elements* if summability of a sequence $s_0, s_1, s_2, \cdots$ to σ implies that, for each complex constant c, the sequence $c, s_0, s_1, s_2, \cdots$ is also summable to σ. It was proved by Knopp, [8], pp. 234-235, that $E(1/2)$ permits adjunction of elements.

THEOREM 4.3. *The transformation* $E(r)$ *permits adjunction of elements if and only if* $|r - 1| < 1$.

Suppose first that $E(r)$ permits adjunction of elements. Then, since the sequence $0, 0, 0, \cdots$ is summable $E(r)$ to 0, the sequence $1, 0, 0, \cdots$ must also be summable $E(r)$ to 0. This implies that $(1 - r)^n \to 0$ as $n \to \infty$ and hence that $|r - 1| < 1$. Suppose now that $|r - 1| < 1$. Let $\sigma_n(r)$ and $\phi_n(r)$ denote respectively the $E(r)$ transforms of the sequence $s_0, s_1, \cdots$ and $a, s_0, s_1, \cdots$. Then, with the aid of the fact that $E^{-1}(r) = E(r^{-1})$, we obtain

$$\phi_n(r) = (1 - r)^n a + \sum_{p=1}^{n} \binom{n}{p} r^p (1 - r)^{n-p} s_{p-1}$$

$$= \theta_n + \sum_{p=0}^{n-1} \binom{n}{p+1} r^{p+1} (1 - r)^{n-p-1} s_p$$

(4.4)

$$= \theta_n + \sum_{p=0}^{n-1} \binom{n}{p+1} r^{p+1} (1 - r)^{n-p-1} \sum_{k=0}^{p} \binom{p}{k} (1/r)^k (1 - 1/r)^{p-k} \sigma_k(r)$$

$$= \theta_n + \sum_{k=0}^{n-1} r(1 - r)^{n-1-k} \sigma_k(r) \sum_{p=k}^{n-1} (-1)^{p-k} \binom{n}{p+1} \binom{p}{k}$$

where $\theta_n \to 0$ as $n \to \infty$. Since [2]

(4.5) $$\sum_{p=k}^{n-1} (-1)^{p-k} \binom{n}{p+1} \binom{p}{k} = 1 \qquad\qquad 0 \leqq k \leqq n - 1,$$

it follows that $\phi_n(r) = \theta_n + \psi_n(r)$ where

(4.6) $$\psi_n(r) = \sum_{k=0}^{n-1} r(1 - r)^{n-1-k} \sigma_k(r).$$

The hypothesis that $|r - 1| < 1$ implies that (4.6) is a regular transformation, from $\sigma_k(r)$ to $\psi_n(r)$, of the form (1.1). Hence $\sigma_n(r) \to \sigma$ implies

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[2] Change of order of summation in the left member leads to the identity

$$\sum_{k=0}^{n-1} \sum_{p=k}^{n-1} (-1)^{p-k} \binom{n}{p+1} \binom{p}{k} z^k = \sum_{k=0}^{n-1} z^k$$

and (4.5) follows.

$\psi_n(r) \to \sigma$ and $\phi_n(r) \to \sigma$. This completes the proof of Theorem 4.3. The condition $|r-1| < 1$ is in fact necessary and sufficient for regularity of (4.6). It follows easily that $E(r)$ permits adjunction of the element 0 if and only if $|r-1| < 1$.

5. Inclusion of $E(r)$ by the generalized Abel method. A series Σu_n with partial sums s_n is said to be summable, by the Abel power series method P, to L if $(1-w)\Sigma s_n w^n$ converges when $|w| < 1$ and $\lim_{w \to 1-}(1-w)\Sigma s_n w^n = L$. The following generalization of this method is due to Silverman and Tamarkin [14]. Let the series and sequence be called summable P^* to L if $(1-w)\Sigma s_n w^n$ converges when $|w|$ is sufficiently small and generates, by analytic extension along radial lines from the origin, a function $p(w)$, analytic over $0 \le w < 1$, such that $\lim_{w \to 1-} p(w) = L$. It was stated, without proof, by Silverman and Tamarkin [14] that $P^* \supset E(r)$ when $0 < r \le 1$.

THEOREM 5.1. *The generalized Abel method P^* includes $E(r)$ if and only if $\Re r > 0$.*

The relation $P^* \supset E(r)$ obviously fails when $r = 0$. Let $\Re r < 0$. Then the center $(1-r^{-1})$ of the circle $C(r)$, in which the sequence z^k is summable $E(r)$, has real part greater than 1. This implies the existence of a real number $z_0 > 1$ such that z_0^n is summable $E(r)$. The function $p(w)$ involved in the definition of P^* summability is in this case

$$(5.2) \qquad p(w) = (1-w)\sum_{k=0}^{\infty} z_0^k w^k = (1-w)/(1-z_0 w), \qquad |w| < 1/z_0$$

and it is clear that analytic extension along radial lines from the origin does not furnish a function $p(w)$ analytic over $0 \le w < 1$. Hence the sequence z_0^n is not summable P^* and accordingly P^* does not include $E(r)$.

Suppose now that $\Re r \ge 0$, $r \ne 0$. Since $E(r)$ has an inverse, each convergent sequence σ_k is the $E(r)$ transform of some sequence s_k; let such a pair of sequences be fixed. Choosing $\delta > 0$ such that the series involved all converge absolutely when $|w| < \delta$, we obtain when $|w| < \delta$ and $w \ne 1$

$$(5.3) \qquad \begin{aligned}(1-w)^{-1}p(w) &= \sum_{n=0}^{\infty} w^n s_n = \sum_{n=0}^{\infty} w^n \sum_{k=0}^{n} \binom{n}{k}(1/r)^k(1-1/r)^{n-k}\sigma_k \\ &= \sum_{k=0}^{\infty}(1/r)^k\sigma_k \sum_{n=k}^{\infty}\binom{n}{k}(1-1/r)^{n-k}w^n \\ &= \sum_{k=0}^{\infty}(w/r)^k\sigma_k \sum_{n=0}^{\infty}\binom{n+k}{k}(w-w/r)^n. \end{aligned}$$

Using the binomial formula for $(1-x)^{-k-1}$, we obtain

(5.4)
$$p(w) = \sum_{k=0}^{\infty} a_k(w)\sigma_k$$

where

(5.41)
$$a_k(w) = \frac{r(1-w)}{r(1-w)+w} \left[\frac{w}{r(1-w)+w}\right]^k.$$

In case $\mathfrak{R}r > 0$, say $\mathfrak{R}r = x > 0$, we find that when $0 < w < 1$

$$|a_k(w)| \leq \frac{|r|(1-w)}{x(1-w)+w} \left[\frac{w}{x(1-w)+w}\right]^k$$

and hence

(5.42)
$$\sum_{k=0}^{\infty} |a_k(w)| \leq |r|/x \qquad\qquad 0 \leq w < 1.$$

Moreover, in this case,

(5.43)
$$\sum_{k=0}^{\infty} a_k(w) = 1, \qquad\qquad 0 \leq w < 1$$

and

(5.44)
$$\lim_{w \to 1-} a_k(w) = 0, \qquad\qquad (k = 0, 1, 2, \cdots).$$

Since the sequence σ_k is convergent and hence bounded, it is now easy to show that the series in (5.4) converges in some open plane set including the segment $0 \leq w < 1$ and that (5.4) furnishes the requisite function $p(w)$ such that $\lim p(w) = \lim \sigma_n$; in completing the argument, we use the fact [4] that the three conditions (5.42), (5.43), and (5.44) ensure regularity of the transformation (5.4). Thus $P^* \supset E(r)$ when $\mathfrak{R}r > 0$. In case $\mathfrak{R}r = 0$ but $r \neq 0$, say $r = iy$ where y is real and $y \neq 0$, then

(5.51)
$$|a_k(w)| = \frac{|y|(1-w)}{[y^2(1-w)^2+w^2]^{1/2}} \left\{\frac{w}{[y^2(1-w)^2+w^2]^{1/2}}\right\}^k,$$

(5.52)
$$\sum_{k=0}^{\infty} |a_k(w)| = \frac{w+[y^2(1-w)^2+w^2]^{1/2}}{|y|(1-w)},$$

and

(5.53)
$$\lim_{w \to 1-} \sum_{k=0}^{\infty} |a_k(w)| = \infty.$$

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[4] For an exposition of, and references to, the subject see Hurwitz [6].

In this case, (5.4) furnishes the function $p(w)$ analytic over $0 \leqq w < 1$; but (5.53) shows that (5.4) is not regular. Therefore, when $\mathcal{R} r = 0$ but $r \neq 0$, $\sigma_n(r) \to \sigma$ does not imply $p(w) \to \sigma$ and accordingly P^* does not include $E(r)$. This completes the proof of Theorem 5.1.

If $\mathcal{R} r > 0$ and $\mathcal{R} q > -1$, then the Euler method $E(r)$ and the Cesàro method $C(q)$ are consistent since, by Theorem 5.1 and the well known fact that the ordinary Abel power series P includes C_q when $\mathcal{R} q > -1$, the generalized Abel method P^* includes both methods.

6. Consistency of the transformations $E(r)$. A family of transformations is said to be *consistent* if no sequence is summable to different values by different transformations of the family. The main result of this section is set forth in the following theorem.

THEOREM 6.1. *The family of transformations $E(r)$ for which $r \neq 0$ is consistent.*

It is a consequence of this theorem that one can define a parameterless method E of summability as follows: A sequence s_n is summable E to σ if a complex number $r \neq 0$ exists such that s_n is summable $E(r)$ to σ.

Consistency of the subfamily of transformations $E(r)$ for which $\mathcal{R} r > 0$ is easily shown by use of Theorem 5.1. If p_1 and q_1 have positive real parts, and s_n is summable $E(p_1)$ to $L(p_1)$ and $E(q_1)$ to $L(q_1)$, then s_n is summable P^* to $L(p_1)$ since $P^* \supset E(p_1)$ and is summable P^* to $L(q_1)$ since $P^* \supset E(q_1)$. Therefore $L(p_1) = L(q_1)$ and consistency of $E(p_1)$ and $E(q_1)$ is established for the case in which $\mathcal{R} p_1 > 0$, $\mathcal{R} q_1 > 0$.

Let p and q be complex numbers not 0. The transformations $E(p)$ and $E(q)$ are consistent if and only if the hypotheses $x_n = E(p)s_n$, $y_n = E(q)s_n$, $x_n \to x$ and $y_n \to y$ imply that $x = y$. Since $E^{-1}(p) = E(p^{-1})$ and $E(q)E^{-1}(p) = E(q/p)$, the hypotheses hold for some sequence s_n if and only if $y_n = E(q/p)x_n$, $x_n \to x$ and $y_n \to y$. Thus $E(p)$ and $E(q)$ are consistent if, and only if, the hypotheses $y_n = E(q/p)x_n$, $x_n \to x$ and $y_n \to y$ imply $x = y$, that is, if, and only if, $E(q/p)$ is consistent with convergence. Likewise, if α is a constant not 0, then $E(\alpha p)$ and $E(\alpha q)$ are consistent if, and only if, $E(q/p)$ is consistent with convergence and hence if, and only if, $E(p)$ and $E(q)$ are consistent.

Suppose now that p and q are such that q/p is not both real and negative; this means that p and q are interior points of a half-plane whose edge is a line through the origin. It is then possible to choose a number α of the form

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$e^{i\phi}$, where ϕ is real, such that the points $p_1 \equiv \alpha p$ and $q_1 \equiv \alpha q$ have positive real parts. It then follows that $E(\alpha p)$ and $E(\alpha q)$ are consistent and hence that $E(p)$ and $E(q)$ are consistent.

It remains for us to prove the following lemma.

LEMMA 6.2. *If p/q is real and negative, then $E(p)$ and $E(q)$ are consistent.*[5]

Our proof of Lemma 6.2 gives, without added complication, a proof of the following theorem.

THEOREM 6.3. *If p/q is real and negative, then each sequence x_n for which the $E(p)$ and $E(q)$ transforms are both bounded must be a constant sequence, that is, a sequence in which each element is equal to the first element.*

That Theorem 6.3 implies Lemma 6.2 is a consequence of the fact that if x_n is summable $E(p)$ and $E(q)$ then the $E(p)$ and $E(q)$ transforms must be bounded, and the fact that a constant sequence is summable to the value of its elements by each transformation $E(r)$. It is a consequence of Theorem 6.3 that if p/q is real and negative, then the constant sequences constitute the intersections of the convergence fields $E(p)$ and $E(q)$. Our proof of Theorem 6.3 is accomplished by proving two lemmas which justify application of a theorem on entire functions due to S. Bernstein [2].

LEMMA 6.4. *Let r be a complex number not 0, let s_n be a sequence of complex numbers, and let d_n and σ_n denote, respectively, the sequence of differences and the $E(r)$ transform of s_n so that*

$$(6.41) \qquad d_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} s_k,$$

$$(6.42) \qquad \sigma_n = \sum_{k=0}^{n} \binom{n}{k} r^k (1-r)^{n-k} s_k.$$

If the transform σ_n is bounded, then there exists a function $f(t)$ analytic at least in the half plane $\Re(t/r) < \frac{1}{2}$ and such that

$$(6.43) \qquad f(t) = \sum_{n=0}^{\infty} d_n t^n$$

---

[5] It is possible to use properties of $E(r)$ to show that Lemma 6.2 will follow if it is shown that $E(-1)$ and $E(1)$ are consistent. The author is indebted to Professor W. A. Hurwitz who worked with him to settle the crucial question whether $E(-1)$ and $E(1)$ are consistent. The proof of this lemma and the following theorem is largely due to Hurwitz.

at least in the circle $|t/r| < \frac{1}{2}$. *Moreover the series in the right member of the equality*

$$(6.44) \qquad g(t) = \sum_{n=0}^{\infty} (d_n/n!)t^n$$

converges for all values of t *and the entire function* $g(t)$ *which it defines is bounded over the set of values of* t *for which* t/r *is real and less than or equal to* 0.

Using (6.41) and (6.42), we find that

$$
\begin{aligned}
d_n &= \sum_{j=0}^{n} (-1)^j \binom{n}{j} \sum_{k=0}^{j} \binom{j}{k} (1/r)^k (1-1/r)^{j-k} \sigma_k \\
(6.45) \qquad &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (1/r)^k \left[\sum_{j=k}^{n} \binom{n-k}{j-k} (1/r-1)^{j-k} \right] \sigma_k \\
&= (1/r^n) \sum_{k=0}^{n} (-1)^k \binom{n}{k} \sigma_k.
\end{aligned}
$$

From (6.45) we obtain, when $|\sigma_n| \leqq M$

$$(6.46) \qquad |d_n| \leqq (M/|r|^n) \sum_{k=0}^{n} \binom{n}{k} = M(2/|r|)^n.$$

This shows that the series in (6.43) converges at least when $|t/r| < \frac{1}{2}$ and that the series in (6.44) converges for all t. When $|t/r| < \frac{1}{2}$, absolute convergence of all of the series involved justifies the computation

$$
\begin{aligned}
(6.47) \quad f(t) &= \sum_{n=0}^{\infty} d_n t^n = \sum_{n=0}^{\infty} \left[(1/r^n) \sum_{k=0}^{n} (-1)^k \binom{n}{k} \sigma_k \right] t^n \\
&= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} (1/r^n)(-1)^k \binom{n}{k} t^n \sigma_k \\
&= \sum_{k=0}^{\infty} (-1)^k (t/r)^k \left[\sum_{n=0}^{\infty} \binom{n+k}{k} (t/r)^n \right] \sigma_k \\
&= \sum_{k=0}^{\infty} (-1)^k (t/r)^k (1-t/r)^{-k-1} \sigma_k = \frac{1}{1-t/r} \sum_{k=0}^{\infty} (-1)^k \left(\frac{t/r}{1-t/r} \right)^k \sigma_k.
\end{aligned}
$$

Since σ_n is bounded, the last member of (6.47) is, as a function of t, analytic over the set of values of t for which

$$|t/r| < |1-t/r|,$$

that is, the set of values of t for which $\mathscr{R}(t/r) < \frac{1}{2}$. This establishes the properties of $f(t)$. For all values of t, use of (6.44) and (6.45) gives

$$(6.48) \quad g(t) = \sum_{n=0}^{\infty} (d_n/n!) t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \frac{1}{k!(n-k)!} (t/r)^n \sigma_k$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \sum_{n=k}^{\infty} \frac{(t/r)^n}{(n-k)!} \sigma_k = e^{t/r} \sum_{k=0}^{\infty} \frac{(-t/r)^k}{k!} \sigma_k.$$

It follows that if $|\sigma_k| \leq M$ and if t/r is real and less than or equal to zero, then

$$(6.49) \qquad\qquad |g(t)| \leq M e^{t/r} \sum_{k=0}^{\infty} \frac{(-t/r)^k}{k!} = M.$$

This completes the proof of Lemma 6. 4.

LEMMA 6. 5. *Let p and q be complex numbers for which q/p is real and negative. Let x_n be a sequence having bounded $E(p)$ and $E(q)$ transforms, and let d_n be the sequence (6.41) of differences of the sequence x_n. Then the functions $f(t)$ and $g(t)$ defined by*

$$(6.51) \qquad\qquad f(t) = \sum_{n=0}^{\infty} d_n t^n, \quad g(t) = \sum_{n=0}^{\infty} (d_n/n!) t^n$$

are entire functions, and $g(t)$ is bounded over the set of values of t for which t/p is real.

Applications of the first part of Lemma 6. 4 with $r = p$ and with $r = q$ show that $f(t)$ is analytic over the half planes H_1 and H_2 of values of t for which $\mathcal{R}(t/p) < \frac{1}{2}$ and $\mathcal{R}(t/q) < \frac{1}{2}$. Since p/q is real and negative, the union of H_1 and H_2 covers the complex plane. Hence $f(t)$ is an entire function, and the first series in (6. 51) must converge for all t. Applications of the second part of Lemma 6. 4 with $r = p$ and with $r = q$ show that $g(t)$ is bounded over the half lines l_1 and l_2 of values of t for which $t/p \leq 0$ and $t/q \leq 0$. Since q/p is real and negative, the half lines l_1 and l_2 constitute the line of values of t for which t/p is real. Therefore $g(t)$ is bounded on this line, and Lemma 6. 5 is proved.

We are now in a position to use the following lemma.[6]

LEMMA 6. 6. *If ρ and M are positive constants, if the sequence*

$$(6.61) \qquad\qquad a_0, a_1 \rho^{-1}, a_2 \rho^{-2}, \cdots$$

is bounded, and if the function $g(z)$ defined by

---

[6] This result of S. Bernstein [2] is stated and proved by Pólya-Szegö [11], vol. 2, p. 35 and pp. 218-219.

$$(6.62) \qquad F(z) = \sum_{n=0}^{\infty} (a_n/n!) z^n$$

is an entire function such that

$$(6.63) \qquad |F(z)| \leqq M$$

for all real values of z, then

$$(6.64) \qquad |F'(z)| \leqq \rho M$$

for all real values of z.

To apply this lemma to prove Theorem 6.3, let $a_n = p^n d_n$ and $z = t/p$. Then, with the notation of Lemmas 6.5 and 6.6, $F(z) = g(t)$ and z is real when t/p is real. Since $f(t)$ is an entire function, the sequence $p^n d_n \rho^{-n}$ ($= a_n \rho^{-n}$) is bounded for each $\rho > 0$. Moreover $|F(z)| \leqq M$ for each real z. Hence, by Lemma 6.6, $|F'(z)| \leqq \rho M$ when $\rho > 0$ and z is real. Therefore $F'(z) = 0$ when z is real. Since $F'(z)$ is an entire function, it follows that $F'(z) = 0$ for all z and that $F(z)$ is a constant. Therefore a_n and d_n must be 0 when $n > 0$. Since solving the equations (6.41) for s_n gives

$$s_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} d_k,$$

it follows that $s_n = d_0$ for each $n = 0, 1, 2, \cdots$ and hence that s_n is a constant sequence. This completes the proof of Theorem 6.3 and hence also that of Theorem 6.1.

If z is a complex number and x_n is the sequence defined by $x_n = z^n + (2 - z)^n$, then $E(-1) x_n = E(1) x_n = x_n$. It is easy to prove directly the fact, implied by Theorem 6.3, that this sequence is bounded if and only if $z = 1$, that is, if, and only if, the sequence is a constant sequence.

7. Series-to-series transformations. If r is a complex number not 0, the formal computation

$$(7.1) \quad \sum_{k=0}^{\infty} u_k = \sum_{k=0}^{\infty} u_k r^{k+1} [1 - (1-r)]^{-k-1} = \sum_{k=0}^{\infty} u_k r^{k+1} \sum_{n=0}^{\infty} \binom{n+k}{k} (1-r)^n$$

$$= \sum_{k=0}^{\infty} u_k r^{k+1} \sum_{n=k}^{\infty} \binom{n}{k} (1-r)^{n-k} = \sum_{n=0}^{\infty} r \sum_{k=0}^{n} \binom{n}{k} r^k (1-r)^{n-k} u_k$$

motivates the definition whereby the series Σu_k is called summable $\mathcal{E}(r)$ to σ if the last series in (7.1) converges to σ; that is, if $\Sigma U_n(r) = \sigma$ where

$$(7.2) \qquad U_n(r) = r \sum_{k=0}^{n} \binom{n}{k} r^k (1-r)^{n-k} u_k.$$

This series-to-series transformation becomes the familiar Euler transformation when $r = \frac{1}{2}$; for discussion and references, see Knopp [8] and Dale [3]. A trivial modification of the computation in (1.5) shows that $\mathcal{E}(p)\mathcal{E}(q) = \mathcal{E}(pq)$.

Setting $V_n = U_0 + U_1 + \cdots + U_n$, $(n = 0, 1, 2, \cdots)$, we see that Σu_n is summable $\mathcal{E}(r)$ to σ if and only if $V_n \to \sigma$ where

$$(7.3) \qquad V_n = \sum_{j=0}^{n} r \sum_{k=0}^{j} \binom{j}{k} r^k (1-r)^{j-k} u_k.$$

Reversal of the order of summation gives

$$(7.4) \qquad V_n = \sum_{k=0}^{n} \Big[\sum_{j=k}^{n} \binom{j}{k} r^{k+1} (1-r)^{j-k} \Big] u_k.$$

Except for differences in notation [7] (7.4) is the series-to-sequence transformation $\mathcal{E}(r)$ obtained by Dale [3] by a different process. If we set

$$s_k = u_0 + \cdots + u_k, \quad u_k = s_k - s_{k-1} \qquad (k = 0, 1, 2, \cdots),$$

where $s_{-1} = 0$, and let

$$(7.5) \qquad b_{nk}{}^{(r)} = \sum_{j=k}^{n} \binom{j}{k} r^{k+1} (1-r)^{j-k},$$

then $\mathcal{E}(r)$ takes the form of a sequence-to-sequence transformation

$$(7.6) \qquad V_n = \sum_{k=0}^{n} a_{nk}{}^{(r)} s_k$$

where

$$(7.61) \qquad a_{nk}{}^{(r)} = b_{nk}{}^{(r)} - b_{n,k+1}{}^{(r)};$$

the sequence s_k is summable $\mathcal{E}(r)$ to σ if $V_n \to \sigma$ as $n \to \infty$. Using (7.61) and (7.5), we find

$$a_{nk}{}^{(r)} = r^{k+1} \Big[\sum_{j=k}^{n} \binom{j}{k} (1-r)^{j-k} + \sum_{j=k+1}^{n} \binom{j}{k+1} (1-r-1)(1-r)^{j-k-1} \Big]$$

$$= r^{k+1} \Big[\sum_{j=k}^{n} \Big\{ \binom{j}{k} + \binom{j}{k+1} \Big\} (1-r)^{j-k} - \sum_{j=k+1}^{n} \binom{j}{k+1} (1-r)^{j-k-1} \Big]$$

$$= r^{k+1} \Big[\sum_{j=k}^{n} \binom{j+1}{k+1} (1-r)^{j-k} - \sum_{j=k+1}^{n} \binom{j}{k+1} (1-r)^{j-k-1} \Big].$$

---

[7] The subscripts in Miss Dale's paper are $1, 2, 3, \cdots$ whereas ours are $0, 1, 2, \cdots$; moreover the parameter r of Miss Dale is the reciprocal of ours.

Cancelling terms from the last sums gives

$$(7.62) \qquad a_{nk}{}^{(r)} = \binom{n+1}{k+1} r^{k+1}(1-r)^{n-k}.$$

Using (7.6) and (7.62), we find that a series $u_0 + u_1 + \cdots$ with partial sums $s_0, s_1, \cdots$ is summable $\mathcal{E}(r)$ to σ if, and only if, $V_n \to \sigma$ where

$$(7.7) \qquad V_{n-1} = \sum_{k=1}^{n} \binom{n}{k} r^k (1-r)^{n-k} s_{k-1}.$$

We are now in a position to prove the following theorem relating the methods $\mathcal{E}(r)$ and $E(r)$.

THEOREM 7.8. If $r \neq 0$, then $\mathcal{E}(r) \subset E(r)$. If $|r-1| < 1$, then $\mathcal{E}(r) \supset E(r)$ while if $r \neq 0$ and $|r-1| \geqq 1$, then $\mathcal{E}(r)$ fails to include $E(r)$.

Suppose $r \neq 0$ and that the sequence $s_0, s_1, s_2, \cdots$ is summable $\mathcal{E}(r)$ to σ. Then (7.7) shows that the sequence $0, s_0, s_1, s_2, \cdots$ is summable $E(r)$ to σ. Since $E(r)$ permits omission of elements, the sequence $s_0, s_1, s_2, \cdots$ is also summable $E(r)$ to σ. This shows that $\mathcal{E}(r) \subset E(r)$. Suppose now that $|r-1| < 1$ and that the sequence $s_0, s_1, \cdots$ is summable $E(r)$ to σ. Then, since $E(r)$ permits adjunction of elements, the sequence $0, s_0, s_1, \cdots$ is summable $E(r)$ to σ. Hence (7.7) implies that the sequence $s_0, s_1, \cdots$ is summable $\mathcal{E}(r)$ to σ. Suppose finally that $r \neq 0$ and $|r-1| \geqq 1$. Then, by Theorem 4.3, there is a sequence $s_0, s_1, s_2, \cdots$ summable $E(r)$ to σ such that the sequence $0, s_0, s_1, \cdots$ is not summable $E(r)$ to σ. Using (7.7), we see that this sequence is not summable $\mathcal{E}(r)$ to σ. Thus $\mathcal{E}(r)$ fails to include $E(r)$.

It is a corollary of Theorem 7.8 that $\mathcal{E}(r)$ and $E(r)$ are equivalent if and only if $|r-1| < 1$. This equivalence was proved by Dale [3] for the case $0 < r < 1$.

It is a corollary of Theorems 6.1 and 7.8 that the methods $\mathcal{E}(r)$ for which $r \neq 0$ are consistent.

For each $r \neq 0$, the $\mathcal{E}(r)$ transform of the geometric series Σz^n is, as given by (7.3), when $z \neq 1$

$$V_n = \sum_{j=0}^{n} r \sum_{k=0}^{j} \binom{j}{k} (rz)^k (1-r)^{j-k} = \sum_{j=0}^{n} r(1-r+rz)^j = \frac{1-(1-r+rz)^{n+1}}{1-z}.$$

Hence Σz^n is summable $\mathcal{E}(r)$ to $1/(1-z)$ if and only if $|1-r+rz| < 1$.

Thus Σz^n is summable $\mathcal{E}(r)$ to $1/(1-z)$ in the same circle $C(r)$ in which the sequence z^n is summable $E(r)$ to 0 and the series Σz^n is summable $E(r)$ to $1/(1-z)$. Therefore, in so far as application to the geometric series Σz^n is concerned, the transformations $\mathcal{E}(r)$ and $E(r)$ are equivalent for each $r \neq 0$.

8. $E(r)$ summability of power series. If, for a fixed $z_0 \neq 0$ and $r \neq 0$, the series $\Sigma c_n z_0{}^n$ is summable $E(r)$, then (2) the series $\Sigma c_n z_0{}^n z^n$ has a positive radius of convergence and accordingly the series $\Sigma c_n z^n$ has a positive radius of convergence. Let $f(z)$ be the function generated by analytic extension, along radial lines from the origin, of the element determined by convergence of $\Sigma c_n z^n$. The open set in which $f(z)$ is thus defined is the *Mittag-Leffler star S*. This star consists of all points of the complex plane not of the form $\rho\zeta$ where $\rho \geq 1$ and ζ is a singular point of $f(z)$. A singular point ζ is a *vertex* of S if $f(z)$ is analytic when z is on the line segment $\theta\zeta$ for which $0 \leq \theta < 1$; the vertices of S belong to the complement of S.

It is easy to see that the power series $\Sigma c_n z^n$ is summable to $f(z)$ by the generalized Abel method P^* when $z \epsilon S$; that $\Sigma c_n z^n$ is in some cases summable P^* and in other cases non-summable P^* when z is a vertex of S; and that $\Sigma c_n z^n$ is non-summable P^* when z is a point in the complement of S which is not a vertex of S. Use of these facts and Theorem 5.1 gives the following theorem.

THEOREM 8.1. *Let $r \neq 0$, let $z_0 \neq 0$, and let the series $\Sigma c_n z^n$ be summable $E(r)$ when $z = z_0$. Then $\Sigma c_n z^n$ has a positive radius of convergence. If $\mathcal{R} r > 0$, then z_0 is either a point or a vertex of the Mittag-Leffler star S. If $\mathcal{R} r > 0$ and $z_0 \epsilon S$, then $f(z_0)$ is the value to which $\Sigma c_n z_0{}^n$ is summable $E(r)$.*

The results of **3** are easily phrased in terms of the geometric series Σz^n which generates the function $1/(1-z)$. It is natural to try to use the Cauchy integral theorem to extend these results to more general power series. Let $\Sigma c_n z^n$ be a power series having a positive finite radius of convergence, and let $f(z)$ and S be defined as above. Corresponding to each vertex ζ of S, let $B(r, \zeta)$ denote the set of points z for which

$$(8.11) \qquad\qquad |z - (1 - r^{-1})\zeta| < |r^{-1}\zeta|.$$

This set $B(r, \zeta)$ is the interior of the circle, with center at the point $(1 - r^{-1})\zeta$, which passes through the point ζ. Let $B(r)$ denote the set of inner points of the intersection of the family of sets $B(r, \zeta)$ determined by the family of vertices ζ of S. This set $B(r)$, which is not a polygon in the ordinary sense,

is (Knopp [8] and Agnew [1]) *the Euler polygon of order r determined by* $\Sigma c_n z^n$. In case r is real and positive, $B(r)$ is always a subset of the interior of the Borel polygon; and the union of the sets $B(r)$ for which $0 < r < 1$ is (Knopp [8] and Rademacher [12]) precisely the interior of the Borel polygon.

The sets $B(r, \zeta)$ and $B(r)$ are determined by the singularities of $f(z)$, being otherwise independent of the coefficients in the series $\Sigma c_n z^n$. In case $f(z)$ has a single singular point ζ, $B(r)$ is the interior of the circle with center at $(1 - r^{-1})\zeta$, which passes through the point ζ. The origin is a point in $B(r)$ if, and only if, $|r - 1| < 1$. The union of the sets $B(r)$ for which $r \neq 0$ is the entire plane with the single point ζ omitted. The union of the sets $B(r)$ for which $|r - 1| < 1$ is the entire plane with the half line $z = \lambda\zeta$, $\lambda \geq 1$, omitted; this union is, accordingly, the Mittag-Leffler star of $f(z)$. Suppose now that $f(z)$ has exactly two singular points, one at $+1$ and the other at -1. If $|r - 1| \geq 1$, the two sets $B(r, \zeta)$ have no points in common and, accordingly, $B(r)$ is empty. If $|r - 1| < 1$, then $B(r)$ is the open non-empty intersection of two open circles each containing the origin. The union of the polygons $B(r)$ for which $|r - 1| < 1$ is, in this case also, the Mittag-Leffler star of $f(z)$. In case $f(z)$ has more than two singular points, the sets $B(r)$ may be less extensive. Suppose, for example, that $f(z)$ has singular points at ± 1 and $\pm i$, and that it has no other singularities. If $|r - 1| \geq 1$, the set $B(r)$ is empty. If $|r - 1| < 1$, the set $B(r)$ is an open set containing the origin. The union of the sets $B(r)$ can be shown (see Theorem 9.1) to consist of the origin and the union of the interiors of the four circles having for diameters the four sides of the square with vertices at ± 1, $\pm i$. This union naturally includes the inner points of the Borel polygon, and is a bounded subset of the Mittag-Leffler star of $f(z)$.

The result of the following theorem was proved by Knopp [8] and Rademacher [12] for the case in which $r = 2^{-p}$, $(p = 1, 2, \cdots)$.

THEOREM 8.2.   *If $|r - 1| < 1$ and $\Sigma c_n z^n$ has a positive finite radius of convergence, then $\Sigma c_n z^n$ is summable $E(r)$ when $z \epsilon B(r)$ and is non-summable $E(r)$ when z is not in the closure of $B(r)$.*

We show first that $\Sigma a_n z^n$ is summable $E(r)$ when $z \epsilon B(r)$. If $z = 0$, then $\Sigma a_n z^n$ is easily shown to be summable $E(r)$ to a_0. Let $z_1 \epsilon B(r)$ and $z_1 \neq 0$. Then $z_1 \epsilon B(r, \zeta)$ so that

$$(8.21) \qquad\qquad |z - (1 - r^{-1})\zeta| < |r^{-1}\zeta|$$

when $z = z_1$ and ζ is a vertex of the star S. If ζ' is a point not in the star

and ζ' is not a vertex, then a vertex ζ and a number $\rho > 1$ exist such that $\zeta' = \rho\zeta$. The circular set of points z' for which

$$(8.22) \qquad\qquad |z' - (1 - r^{-1})\zeta'| < |r^{-1}\zeta'|$$

includes the set of z for which (8.21) holds and hence includes z_1. It follows that the circular set of points u for which

$$(8.23) \qquad\qquad |z_1 - (1 - r^{-1})u| = |r^{-1}u|$$

lies in the star. This means that, when $\theta = 1$, the circular set of points u for which

$$(8.24) \qquad\qquad |rz_1 u^{-1} - (r - 1)| = \theta$$

lies in the star. When $\theta > |r - 1|$, the equation of the circle (8.24) can be written in the form

$$(8.25) \qquad \left| \frac{u}{rz_1} - \frac{1 - \bar{r}}{\theta^2 - |r - 1|^2} \right| = \frac{\theta}{\theta^2 - |r - 1|^2}.$$

Since the center and radius of the circle are, as functions of the real variable θ, continuous at $\theta = 1$, and since moreover the star is an open point set, we can fix θ such that $|r - 1| < \theta < 1$ and the circle defined by (8.24) lies in the star. It is easily verified that the points 0 and z_1 are interior points of the circle; this gives the following lemma which we state for future reference.

LEMMA 8.3. *If $|r - 1| < 1$ and $z_1 \epsilon B(r)$, then θ can be fixed such that $|r - 1| < \theta < 1$ and the circle of points u for which*

$$(8.31) \qquad\qquad |rz_1 u^{-1} + 1 - r| = \theta$$

lies in the star and contains the points 0 and z_1 in its interior.

Let C be the circle of points u for which (8.31) holds. Then, by the Cauchy integral formula,

$$(8.32) \qquad c_n = (1/2\pi i) \int_C (f(u)/u^{n+1})\,du \qquad\qquad (n = 0, 1, 2, \cdots).$$

The terms of the $\mathcal{E}(r)$ series-transform ΣU_n of the series $\Sigma c_n z_1^n$ are [see (7.2)] given by

$$U_n = r \sum_{k=0}^{n} \binom{n}{k} r^k (1-r)^{n-k} c_k z_1{}^k$$

$$(8.33) \qquad = (r/2\pi i) \int_C (f(u)/u) \sum_{k=0}^{n} \binom{n}{k} (rz_1/u)^k (1-r)^{n-k} du$$

$$= (r/2\pi i) \int_C (f(u)/u) (rz_1 u^{-1} + 1 - r)^n du.$$

Using (8.33) and (8.31) we obtain

$$(8.34) \qquad |U_n| \leqq \theta^n (|r|/2\pi) \int_C (|f(u)|/|u|) |du|$$

and, since $0 < \theta < 1$, $\Sigma |U_n| < \infty$. This means that $\Sigma c_n z_1{}^n$ is summable $\mathcal{E}(r)$; but, since $|r-1| < 1$, $E(r)$ and $\mathcal{E}(r)$ are equivalent and hence $\Sigma c_n z_1{}^n$ is summable $E(r)$. This establishes the fact that $\Sigma c_n z^n$ is summable $E(r)$ when $z \, \epsilon \, B(r)$.

We now prove the following theorem which will be applied to complete the proof of Theorem 8.2.

THEOREM 8.4. *If $|r-1| < 1$ and the series Σu_n is summable $E(r)$, then the series $\Sigma u_n z^n$ is summable $E(r)$ for each z for which*

$$(8.41) \qquad |z| + |r-1| \, |z-1| < 1.$$

Moreover the function $f(z)$ generated by $\Sigma u_n z^n$ is analytic over the open circular set of points z for which

$$(8.42) \qquad |z| < |r - rz + z|.$$

Since $|r-1| < 1$, $E(r)$ and $\mathcal{E}(r)$ are equivalent. Hence Σu_n is summable $\mathcal{E}(r)$ and accordingly ΣU_n converges where

$$(8.43) \qquad U_n = r \sum_{k=0}^{n} \binom{n}{k} r^k (1-r)^{n-k} u_k.$$

Dividing (8.43) by r and using the formula for the inverse of $E(r)$ we obtain

$$u_j = r^{-1} \sum_{k=0}^{j} \binom{j}{k} (1/r)^k (1 - 1/r)^{j-k} U_k.$$

For each complex z, the terms of the $\mathcal{E}(r)$ transform $\Sigma U_n(z)$ of the series $\Sigma u_j z^j$ are given by

$$U_n(z) = r \sum_{j=0}^{n} \binom{n}{j} r^j (1-r)^{n-j} z^j r^{-1} \sum_{k=0}^{j} \binom{j}{k} (1/r)^k (1-1/r)^{j-k} U_k.$$

Reversing the order of summation and simplifying the result we find

(8.44) $$U_n(z) = \sum_{k=0}^{n} \binom{n}{k} z^k (1-r+rz-z)^{n-k} U_k.$$

The convergence of ΣU_k implies the existence of a constant M such that $|U_k| \leq M$ for each $k = 0, 1, \cdots$. Hence (8.44) implies that

(8.45) $$|U_n(z)| \leq M[|z| + |1-r+rz-z|]^n.$$

It follows that $\Sigma U_n(z)$ converges and that $\Sigma u_n z^n$ is summable $\mathcal{E}(r)$ and hence also $E(r)$ when (8.41) holds. Applying Theorem 8.1, we see that $\Sigma u_n z^n$ is summable $E(r)$ and $\mathcal{E}(r)$ to $f(z)$ when (8.41) holds. Hence, when (8.41) holds, use of (8.44) gives

$$f(z) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} z^k (1-r+rz-z)^{n-k} U_k.$$

When $|z|$ is sufficiently small, this series converges absolutely and reversal of the order of summation gives

(8.46) $$f(z) = \sum_{k=0}^{\infty} \frac{z^k}{(r-rz+z)^{k+1}} U_k.$$

Using again the fact that $|U_k| \leq M$, we see that the right member of (8.46) and hence $f(z)$ are analytic over the open set of points z for which (8.42) holds. This proves Theorem 8.4.

We are now in a position to complete the proof of Theorem 8.2 by showing that if $|r-1| < 1$ and $\Sigma c_n z^n$ is summable $E(r)$ when $z = z_1$, then $z_1 \epsilon \overline{B}(r)$. Using Theorem 8.4 with $u_n = c_n z_1^n$, we see that $\Sigma c_n z_1^n t^n$ generates a function $g(t)$ analytic when

$$|t| < |r-rt+t|.$$

Setting $z = z_1 t$ we see that the function $f(z)$ generated by $\Sigma c_n z^n$ is analytic when

(8.47) $$|z_1 - (1-r^{-1})z| > |r^{-1}z|.$$

Therefore (8.47) fails to hold when z is a vertex ζ of the star; that is,

(8.48) $$|z_1 - (1-r^{-1})\zeta| \leq |r^{-1}\zeta|$$

when ζ is a vertex. It follows that the line segment $z = \rho z_1$, $0 \leqq \rho < 1$, lies in the open circular set $B(r, \zeta)$ and hence also in the intersection $B(r)$. Therefore $z_1 \epsilon \overline{B(r)}$ and the proof of Theorem 8. 2 is complete.

We show, by an example, that the conclusion of Theorem 8. 2 may fail when r is a complex number for which $|r-1| > 1$. For each $r \neq 0$, the Euler polygon $B(r)$ determined by the series

$$(8. 5) \qquad 1 + 0 + z^2 + z^3 + z^4 + z^5 + \cdots$$

is the non-empty open circular set of points z for which

$$| z - (1 - r^{-1}) | < | r^{-1} |.$$

Let r be fixed such that $r \neq 0$ and $|r-1| \geqq 1$. Then the points 0 and 1 are not in $B(r)$. When $z \epsilon B(r)$, the $E(r)$ transform of the series (8. 5) is given by

$$\sigma_n = \frac{1}{1-z} - z + z(1-r)^n - \frac{z}{1-z} (rz + 1 - r)^n$$

and it is easy to show that $\lim \sigma_n$ fails to exist. Thus, in this case, $B(r)$ is non-empty and the series is non-summable $E(r)$ for each $z \epsilon B(r)$. It may be noted that the series $1 + z + z^2 + \cdots$ is summable $E(r)$ to $1/(1-z)$ for each $z \epsilon B(r)$, and that the series $0 + z + 0 + 0 + \cdots$ is nonsummable $E(r)$ for each $z \epsilon B(r)$.

9. The union $\mathcal{B}$ of the sets $B(r)$ for which $\mathcal{R} r > 0$. Let $\Sigma c_n z^n$ be a series having a finite positive radius of convergence and let $B(r)$ be defined as in the previous section. Let $\mathcal{B}$ denote the union of the sets $B(r)$ for which $\mathcal{R} r > 0$. The set $\mathcal{B}$, being the union of open subsets of the Mittag-Leffler star S, is an open subset of S. If $\mathcal{R} r_1 > 0$, then it is possible to choose numbers r_2 and λ such that $|r_2-1| < 1$, $\lambda > 1$, and $r_1 = \lambda r_2$. It then follows that, for each singular point ζ, $B(r_1, \zeta)$ is a subset of $B(r_2, \zeta)$ and hence that $B(r_1)$ is a subset of $\mathcal{B}(r_2)$. Therefore B may be otherwise described as the union of all sets $B(r)$ for which $|r-1| < 1$.

Our interest in the set $\mathcal{B}$ lies in the fact that if $z_1 \epsilon \mathcal{B}$ then $\Sigma c_i z_1^n$ is summable $E(r)$ for some r for which $|r-1| < 1$, and if z_2 is not in the closure $\overline{\mathcal{B}}$ of $\mathcal{B}$ then $\Sigma c_n z_2^n$ is nonsummable $E(r)$ for each r for which $\mathcal{R} r > 0$. The following theorem characterizes the set $\mathcal{B}$ in terms of the singular points of the function $f(z)$ generated by $\Sigma c_n z^n$. The Borel polygon is an intersection of half-planes; the set $\mathcal{B}$ turns out to be a union of circular sets.

THEOREM 9.1. *The union $\mathcal{B}$ of the sets $B(r)$ for which $\mathcal{R}r > 0$ is the union of the sets of inner points of all circles which surround the origin and lie in the Mittag-Leffler star of $\Sigma c_n z^n$.*

The set $\mathcal{B}$ can be otherwise described as the set consisting of the origin alone and of the union of the interiors of the circles which pass through the origin and exclude the singular points.

Let U denote the union of the sets described in the theorem. To show that $\mathcal{B} \subset U$, let z_1 be a point in $\mathcal{B}$. Then r exists such that $|r - 1| < 1$ and $z_1 \in B(r)$. Lemma 8.3 furnishes a circle, in the star, containing 0 and z_1 in its interior. Thus $z_1 \in U$ and hence $\mathcal{B} \subset U$. To show that $U \subset \mathcal{B}$, let z_1 be a point in U. Then, by definition of U, there is a circle C which lies in the star and which contains the points 0 and z_1 in its interior. It is obvious from the definitions of $B(r)$ and $\mathcal{B}$ that if $f_1(z)$ and $f(z)$ are so related that the star of $f_1(z)$ is a subset of the star of $f(z)$, then the sets $B_1(r)$ and $\mathcal{B}_1$ formed for $f_1(z)$ are, respectively, subsets of the sets $B(r)$ and $\mathcal{B}$ formed for $f(z)$. Hence we can prove that $z_1 \in \mathcal{B}$ and complete the proof of Theorem 9.1 by proving the following theorem which is in fact a corollary of Theorem 9.1.

THEOREM 9.2. *If $|r - 1| < 1$, if C is a circle containing the origin in its interior, if $f_1(z)$ is analytic inside C, and if each point of C is a singular point ζ of $f_1(z)$, then the set $\mathcal{B}_1$ is the set interior to C.*

It follows from the part of Theorem 9.1 already proved, and is obvious from the definition of $\mathcal{B}_1$, that the points of $\mathcal{B}_1$ are interior to C. To show that each point z_1 interior to C is a point of $\mathcal{B}_1$, let the radius and center of C be R and Ae^{ia} where $A > 0$ and $-\pi < \alpha \leq \pi$. Choose B and β such that $B > 0$, $-\pi < \beta \leq \pi$ and

(9.21) $$z_1 = Ae^{ia} + Be^{i(a+\beta)}.$$

Since 0 and z_1 are interior to C, we have $A < R$ and $B < R$. The diameter through the origin of the circle C containing the singular points ζ of $f_1(z)$ has its ends at the points $(A + R)e^{ia}$ and $(A - R)e^{ia}$. Hence the circle containing the points ζ^{-1} has the ends of a diameter at the reciprocal points, and it follows easily that

(9.22) $$\frac{1}{\zeta} = -\frac{Ae^{-ia}}{R^2 - A^2} + \frac{R}{R^2 - A^2} e^{i\phi}$$

where $-\pi < \phi \leq \pi$. Using (9.21) and (9.22), we obtain

$$(9.23) \qquad \left| \frac{z_1}{\zeta} + \frac{A^2 + ABe^{i\beta}}{R^2 - A^2} \right| = \frac{R \left| A + Be^{i\beta} \right|}{R^2 - A^2}.$$

Let r be defined by the formula

$$(9.24) \qquad r = (R^2 - A^2)/(R^2 + ABe^{i\beta}).$$

Then

$$(1 - r^{-1}) = - (A^2 + ABe^{i\beta})/(R^2 - A^2)$$

and

$$\frac{R \left| A + Be^{i\beta} \right|}{R^2 - A^2} < \frac{\left| R^2 + ABe^{i\beta} \right|}{R^2 - A^2} = \frac{1}{|r|}$$

so that (9.23) gives

$$\left| z_1/\zeta - (1 - r^{-1}) \right| < \left| r^{-1} \right|$$

and hence

$$(9.25) \qquad \left| z_1 - (1 - r^{-1})\zeta \right| < \left| r^{-1}\zeta \right|.$$

Thus $z_1 \epsilon B_1(r, \zeta)$ for each ζ and accordingly $z_1 \epsilon B_1(r)$. Since (9.24) and the inequalities $0 < A < R$, $0 < B < R$ imply that $|r - 1| < 1$ and $\mathcal{R} r > 0$, we conclude that $z_1 \epsilon \mathcal{B}_1$ and the proof of Theorems 9.2 and 9.1 is complete.

If the center of the circle C of Theorem 9.2 is at the origin, then the set B_1 and the interior B of the Borel polygon each coincides with the interior of C. If the center is not at the origin, B_1 still coincides with the interior of C; but the Borel polygon is now an ellipse, inscribed in the circle, with center at the center of the circle and one focus at the origin. (Proof of the latter fact is a straightforward exercise in finding an envelope.) Accordingly, Euler methods $E(r)$ for which $|r - 1| < 1$ evaluate $\Sigma c_n z^n$ to $f_1(z)$ throughout the interior of C, but the regular methods $E(r)$, for which $0 < r \leqq 1$, cannot evaluate $\Sigma c_n z^n$ outside the ellipse in C. In case the origin is near the circumference of C (as compared with its distance from the center of C) the ellipse is flat and includes a small proportion of the area of C.

10. Transformations $E(r_n)$. Corresponding to each sequence $r_0, r_1, \cdots$ of complex numbers, let $E(r_n)$ denote the transformation

$$(10.1) \qquad \sigma_n = \sum_{k=0}^{n} \binom{n}{k} r_n^k (1 - r_n)^{n-k} s_k$$

by means of which a sequence $s_0, s_1, \cdots$ is summable $E(r_n)$ to σ if $\sigma_n \to \sigma$ as

$n \to \infty$. The main problem involving transformations $E(r_n)$ which we consider here is that of characterizing those for which the numbers $r_0, r_1, \cdots$ are positive and $E(r)$ includes all regular Euler methods.

THEOREM 10. 2. *If $r_n > 0$ for each n, then $E(r_n)$ includes $E(r)$ for each r in the interval $0 < r \leqq 1$ if, and only if, $r_n \to 0$ and $nr_n \to \infty$.*[3]

If y_n and x_n are, respectively, the $E(r_n)$ and $E(r)$ transforms of a sequence s_n, then use of the formula for the inverse of $E(r)$ gives

$$ y_n = \sum_{p=0}^{n} \binom{n}{p} r_n{}^p (1 - r_n)^{n-p} \sum_{k=0}^{p} \binom{p}{k} (1/r)^k (1 - 1/r)^{p-k} x_k. $$

Reversing the order of summation and simplifying the result we obtain

(10. 3) $$ y_n = \sum_{k=0}^{n} \binom{n}{k} (r_n/r)^k (1 - r_n/r)^{n-k} x_k. $$

This means that the transformation $E(r_n) E^{-1}(r)$ has the form

(10. 4) $$ y_n = \sum_{k=0}^{n} a_{nk}{}^{(r)} x_k $$

where

(10. 5) $$ a_{nk}{}^{(r)} = \binom{n}{k} (r_n/r)^k (1 - r_n/r)^{n-k}; $$

and that $E(r_n) \supset E(r)$ if, and only if, this transformation is regular. The condition

(10. 6) $$ \sum_{k=0}^{n} a_{nk}{}^{(r)} = 1 \qquad\qquad (n = 0, 1, 2, \cdots) $$

is satisfied for each r without restriction on the sequence r_n. Since $r_n > 0$ and

(10. 7) $$ \sum_{k=0}^{n} |a_{nk}{}^{(r)}| = (|r_n/r| + |1 - r_n/r|)^n, $$

it is easy to show that the condition (1. 41) is satisfied for each $r > 0$ if, and

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[3] An obvious modification shows that, when a real angle ϕ is fixed and $r_0, r_1, \cdots$ are positive numbers, the transformation $E(r_n e^{i\phi})$ includes $E(r e^{i\phi})$ for each $r > 0$ if, and only if, $r_n \to 0$ and $nr_n \to \infty$ as $n \to \infty$.

only if, $r_n \to 0$. Suppose now that $E(r_n) \supset E(r)$ when $0 < r \leqq 1$. Then $r_n \to 0$ and $E(r_n) \supset E(1)$ so that $a_{no}{}^{(r)} \to 0$ as $n \to \infty$. Since

$$a_{n,0}{}^{(1)} = (1 - r_n)^n = [(1 - r_n)^{1/r_n}]^{nr_n} = (e^{-1} + \epsilon_n)^{n/r_n}$$

where $\epsilon_n \to 0$, we conclude easily that $nr_n \to \infty$ as $n \to \infty$. To complete the proof of Theorem 10.2, suppose $r_n > 0$, $r_n \to 0$, and $nr_n \to \infty$; we have to show that

(10.8) $$\lim_{n \to \infty} a_{nk}{}^{(r)} = 0 \qquad (k = 0, 1, 2, \cdots)$$

when $0 < r < 1$. The result is established with the aid of the computation

$$a_{nk} = (1 - r_n/r)^{-k}(1/r)^k \binom{n}{k} r_n{}^k [(1 - r_n/r)^{1/r_n}]^{nr_n} = A_n (nr_n)^k [e^{-1/r} + \epsilon_n]^{nr_n},$$

in which A_n is a bounded sequence and $\epsilon_n \to 0$ as $n \to \infty$, and Theorem 10.2 is proved.

It is a consequence of Theorem 10.2 that, when $r_n > 0$, $r_n \to 0$, and $nr_n \to \infty$, the transformation $E(r_n)$ is a regular sequence-to-sequence transformation with a triangular matrix which evaluates each power series $\Sigma c_n z^n$ at each point inside the Borel polygon. It would be interesting to know how these transformations $E(r_n)$ are related to each other and to other methods of summability. It is readily seen that, unlike two transformations of the form $E(r)$, two transformations of the form $E(r_n)$ do not necessarily commute. When $r_n \neq 0$ for each n, $E(r_n)$ has an inverse; but the inverse is not necessarily of the form $E(q_n)$.

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ON SEQUENCES WITH VANISHING EVEN OR ODD DIFFERENCES.*

By Ralph Palmer Agnew.

1. Introduction. Let $x_0, x_1, x_2, \cdots$ be a sequence of complex numbers and let

$$(1) \qquad d_n \equiv \Delta^n x_0 \equiv \sum_{k=0}^{n} (-1)^k \binom{n}{k} x_k \qquad (n = 0, 1, 2, \cdots)$$

denote its sequence of differences. It is the object of this note to show that the following two theorems are corollaries of Theorem 6.3 of the preceding paper.

Theorem 1. *If x_n is a bounded sequence whose even differences all vanish, that is, if*

$$d_0 = d_2 = d_4 = \cdots = 0,$$

then $x_n = 0$ for each $n = 0, 1, 2, \cdots$.

Theorem 2. *If x_n is a bounded sequence whose odd differences all vanish, that is, if*

$$d_1 = d_3 = d_5 = \cdots = 0,$$

then $x_n = x_0$ for each $n = 1, 2, 3, \cdots$.

2. Proof of the theorems. Let x_n be a bounded sequence. Use of (1) gives

$$(2) \quad \sum_{n=0}^{\infty} (-1)^n (d_n/n!) t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^{n-k}}{k!\,(n-k)!}\, t^n x_k$$
$$= \sum_{k=0}^{\infty} (t^k/k!) \left[\sum_{n=k}^{\infty} \frac{(-t)^{n-k}}{(n-k)!} \right] x_k = e^{-t} \sum_{k=0}^{\infty} (x_k/k!) t^k,$$

the computation being justified by the absolute convergence of the series. If

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$d_n = 0$ when n is even (odd), then the members of (2) must be odd (even) functions of t and accordingly

$$(3) \qquad\qquad e^{-t} \sum_{n=0}^{\infty} (x_n/n!) t^n = \lambda e^t \sum_{n=0}^{\infty} (x_n/n!)(-t)^n$$

where $\lambda = -1$ $(\lambda = +1)$.   Hence

$$(4) \quad \sum_{n=0}^{\infty} (x_n/n!) t^n = \lambda \sum_{a=0}^{\infty} ((2t)^a/a!) \sum_{k=0}^{\infty} (x_k/k!)(-t)^k$$

$$= \lambda \sum_{n=0}^{\infty} \sum_{a+k=n} \frac{(2t)^a}{a!} \frac{x_k}{k!} (-t)^k = \lambda \sum_{n=0}^{\infty} (t^n/n!) \sum_{k=0}^{n} \binom{n}{k} (-1)^k 2^{n-k} x_k.$$

Equating coefficients of t^n, we obtain

$$x_n = \lambda \sum_{k=0}^{n} \binom{n}{k} r^k (1-r)^{n-k} x_k$$

where $r = -1$. Thus the $E(-1)$ transform of the sequence x_n is, except for the factor λ, the sequence x_n itself. Thus x_n has bounded $E(1)$ and $E(-1)$ transforms. Therefore, by Theorem 6.3 of the previous paper, $x_0 = x_1 = x_2 = \cdots$.   In case $d_0 = 0$, we have $x_0 = d_0 = 0$.

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